CSI 5165 Combinatorial Algorithms — Fall 2015
Computer Science — University of Ottawa

Homework Assignment #1 (100 points, weight 8%)
Due: October 19, 11:59PM, via blackboard (uottawa virtual campus).
Your presence in class on October 19 is required for the assignment to be accepted the same day.
Do not miss class to finish an assignment; instead request a 3hs extension which I can grant.

Generating elementary combinatorial objects

1. (35 points) **Generalized Lexicographical n-tuples**

   When we generated all subsets of an n-set in lexicographical order of their characteristics vectors, our algorithms were really generating binary n-tuples in lexicographical ordering. We noticed that the successor algorithm was equivalent to adding 1 to an n-bit binary number; the ranking algorithm was nothing more than transforming a given n-tuple (considered as a binary representation of a number) to its numerical representation; and the unranking algorithm was equivalent to transforming a number into an n-tuple corresponding to its binary representation.

   In this exercise, you will develop similar algorithms, but we are not going to be looking at binary n-tuples, but a tuple in some "mixed-basis". Consider fixed $m_1, m_2, \ldots, m_n$, where $m_i \geq 1$, for all $1 \leq i \leq n$. We will be generating in lexicographical order all n-tuples $(a_1, a_2, \ldots, a_n)$ where each component a_i satisfies $0 \leq a_i < m_i$, for all $1 \leq i \leq n$. The case of binary n-tuples (subsets of an n-set) is a special case of this where $m_i = 2$ for all $1 \leq i \leq n$.

   For example, for $(m_1, m_2, m_3) = (3, 4, 2)$ the 24 tuples in lexicographic order are: 000, 001, 010, 011, 020, 021, 030, 031, 100, 101, 110, 111, 120, 121, 130, 131, 200, 201, 210, 211, 220, 221, 230, 231.

   (a) Give successor, ranking and unranking algorithms (pseudocode is fine); let's name these algorithms tupleSuccessor, tupleRanking and tupleUnranking, respectively.

   (b) Solve a related problem where there are lower and upper bounds for the components. In other words, let $l_1, l_2, \ldots, l_n$ and $u_1, u_2, \ldots, u_n$ be fixed integers with $0 \leq l_i \leq u_i$, for all $1 \leq i \leq n$. We must generate, in lexicographical order, n-tuples $(a_1, a_2, \ldots, a_n)$ so that $l_i \leq a_i \leq u_i$, for all $1 \leq i \leq n$. The binary example had $l_i = 0$ and $u_i = 1$, for all $1 \leq i \leq n$. Write boundedTupleSuccessor, boundedTupleRanking and boundedTupleUnranking; your algorithms may call the corresponding algorithms given in part 1a.

2. (30 points) **Gray codes and revolving door ordering**

   Suppose $1 \leq k \leq n$, and we delete all vectors in the binary reflected Gray code G^n that do not correspond to subsets of cardinality k. Now we convert the weight-k binary vectors to k-subsets by associating the vector $(a_n, \ldots, a_2, a_1)$ to the k-set $\{i : a_i = 1\}$ (note that this is reversed with respect to the textbook presentation that associates a_1 to the presence of element n in the set).

   Prove using induction on n that the vectors that remain, let us call $(G^n)_k$, comprise the vectors listed in the revolving door ordering $A^{n,k}$.

   Example for $n = 3$:

   $G^3 = [000, 001, 011, 010, 110, 111, 101, 100]$

k	$(G^3)_k$	$A^{3,k}$
0	$(G^3)_0 = [000]$	$A^{3,0} = [\{\}]$
1	$(G^3)_1 = [001, 010, 100]$	$A^{3,1} = [\{1\}, \{2\}, \{3\}]$
2	$(G^3)_2 = [011, 110, 101]$	$A^{3,2} = [\{1, 2\}, \{2, 3\}, \{1, 3\}]$
3	$(G^3)_3 = [111]$	$A^{3,3} = [\{1, 2, 3\}]$

3. (35 points) **Signed permutations**
A signed permutation is a permutation with optional signs attached to the elements; therefore, there are $2^n n!$ such signed permutations of n elements. For example $31\overline{2}$ is a signed permutation. Give an algorithm that generates all signed permutations of $\{1, 2, \ldots, n\}$ where each step either interchanges two adjacent elements or negates the first element. This resembles the Trotter-Johnson minimal change ordering for permutations.

Some examples of this order are: for $n = 1$ we have $[1, \overline{1}]$ and for $n = 2$ we have $[12, 21, \overline{2}1, 1\overline{2}, \overline{1}\overline{2}, \overline{2}\overline{1}, 2\overline{1}, \overline{1}2]$.

$$\begin{array}{l}
123, 132, 312, \overline{3}12, 1\overline{3}2, 12\overline{3}, \\
21\overline{3}, 2\overline{3}1, \overline{3}21, 321, 231, 213, \\
\overline{2}13, \overline{2}31, 3\overline{2}1, \overline{3}\overline{2}1, \overline{2}\overline{3}1, \overline{2}1\overline{3}, \\
1\overline{2}\overline{3}, 1\overline{3}\overline{2}, \overline{3}1\overline{2}, 31\overline{2}, 13\overline{2}, 1\overline{2}3, \\
\overline{1}\overline{2}3, \overline{1}3\overline{2}, 3\overline{1}\overline{2}, \overline{3}\overline{1}\overline{2}, \overline{1}\overline{3}\overline{2}, \overline{1}\overline{2}\overline{3}, \\
\overline{2}\overline{1}\overline{3}, \overline{2}\overline{3}\overline{1}, \overline{3}\overline{2}\overline{1}, 3\overline{2}\overline{1}, \overline{2}3\overline{1}, \overline{2}\overline{1}3, \\
2\overline{1}3, 23\overline{1}, 32\overline{1}, \overline{3}2\overline{1}, 2\overline{3}\overline{1}, 2\overline{1}\overline{3}, \\
\overline{1}2\overline{3}, \overline{1}\overline{3}2, \overline{3}\overline{1}2, 3\overline{1}2, \overline{1}32, \overline{1}23,
\end{array}$$

Note that for n you have $2^n n!$ signed permutations, we obtain the ones for $(n+1)$ each of the signed permutations for n will show up $2(n+1)$ times as a nested permutation. For $n+1$ steps the element $n+1$ (with or without sign) moves from the end position until the first position, then its sign gets reversed and it moves from the first position to the last.

Note: It will be useful to study well how the Trotter-Johnson algorithm works for minimal change of regular permutations as it will be helpful here. You may find useful implementing your method to double check its correctness.

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4. **NOT TO BE HANDED IN: practice with combinatorial generation algorithms**
Calculate the result for the following operations. Show your work.

   - Subsets:
     Give the SUCCESSOR and the RANK of 11010110 in the Gray code G^8.
   - k-subsets:
     Give RANK of $\{3, 6, 7, 9\}$ considered as a 4-subset of $\{1, \ldots, 13\}$ in lexicographic and revolving-door order. What is the SUCCESSOR in each of these orders?
   - Permutations:
     Find the rank and successor of the permutation $[2, 4, 6, 7, 5, 3, 1]$ in lexicographic and Trotter-Johnson order.

     UNRANK the rank $r = 54$ as a permutation of $\{1, 2, 3, 4, 5\}$, using the lexicographic and Trotter-Johnson order.