Heuristic Search

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Heuristic Search vs Exhaustive Search

Exhaustive Search

- Backtracking (backtracking with bounding):
  - Find all feasible solutions.
  - Find one optimal solution.
  - Find all optimal solutions.
- Branch-and-Bound:
  - Find one optimal solution.

Heuristic Search

Types of problem it can be applied to:

- Find 1 optimal solution (when optimum value is known)
- Find a "close to" optimal solution (the best solution we manage).

Heuristics methods we will study:

- Hill-climbing, Simulated annealing, Tabu search, Genetic algorithms.

Characteristics of heuristic search

- The state space is not fully explored.
- Randomization is often employed.
- There is a concept of neighbourhood search.
- **Heuristics** are applied to explore the solutions.
  The word "heuristics" means "serving or helping to find or discover" or "proceeding by trial and error".

A general framework for heuristic search

Generic Optimization Problem (maximization):

Instance: A finite set $\mathcal{X}$.
an objective function $P : \mathcal{X} \to Z$.
m feasibility functions $g_j : \mathcal{X} \to Z$, $1 \leq j \leq m$.
Find: the maximum value of $P(X)$
subject to $X \in \mathcal{X}$ and $g_j(X) \geq 0$, for $1 \leq j \leq m$.

Exercise: pick your favorite combinatorial optimization problem and write it in this framework.

A general framework for heuristic search (cont'd)

Designing a heuristic search:

1. Define a **neighbourhood function** $N : \mathcal{X} \to 2^{\mathcal{X}}$.
   E.g. $N(X) = \{X_1, X_2, X_3, X_4, X_5\}$.
2. Design a **neighbourhood search**:
   Algorithm that finds a feasible solution on the neighbourhood of a feasible solution X.
   There are two types of neghbourhood searches:
   - Exhaustive (chooses best profit among neighbour points)
   - Randomized (picks a random point among the neighbour points)

Defining a neighbourhood function

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So, $N(X)$ is a subset of $\mathcal{X}$.

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- Computing $N(X)$ should be fast, and in particular $|N(X)|$ shouldn't be too large.

Examples of neighbourhood functions

First, define $dist(X,Y)$ for $X, Y \in \mathcal{X}$.
Let d_0 be a constant positive integer.
We can define a neighbourhood function as follows:

$$N_{d_0}(X) = \{Y \in \mathcal{X} : dist(X,Y) \leq d_0\}.$$

Examples of neighbourhood functions based on distances

- $\mathcal{X} = \{0,1\}^n$, set of all binary n-tuples.
  Here $dist$ is the Hamming distance.
  $N_1([010]) = \{[000], [110], [011], [010]\}$.

$$|N_{d_0}(X)| = \sum_{i=0}^{d_0} \binom{n}{i}.$$

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- $\mathcal{X} =$ set of all permutations of $\{1,2,\ldots,n\}$.
  Let $\alpha = [\alpha_1,\ldots,\alpha_n]$ and $\beta = [\beta_1,\ldots,\beta_n]$ be two permutations.
  Define distance as follows: $dist(\alpha,\beta) = |\{i : \alpha_i \neq \beta_i\}|$.
  Note that $N_1(X) = \{X\}$ is not very useful; we need $d_0 > 1$.
  $$N_2([1,2,3,4]) = \{[1,2,3,4],[2,1,3,4],[3,2,1,4],\\ [4,2,3,1],[1,3,2,4],[1,4,3,2],[1,2,4,3]\}$$
  $|N_2(X)| = 1 + \binom{n}{2}$.

Designing a neighbourhood search algorithm

Input: X

Output: $Y \in N(X) \setminus \{X\}$ such that Y is feasible, or "fail".

Possible Neighbourhood Search Strategies:

1. Find a feasible $Y \in N(X) \setminus \{X\}$ such that $P(Y)$ is maximized.
   Return "fail" if there is no feasible solution in $N(X) \setminus \{X\}$.
2. Find a feasible $Y \in N(X) \setminus \{X\}$ such that $P(Y)$ is maximized.
   if $P(Y) > P(X)$ then return Y; else return "fail".
   (steepest ascent method)
3. Find any feasible $Y \in N(X) \setminus \{X\}$.
   Return "fail" if there is no feasible solution in $N(X) \setminus \{X\}$.
4. Find any feasible $Y \in N(X) \setminus \{X\}$.
   if $P(Y) > P(X)$ then return Y; else return "fail".

Strategies 1 and 2 may be exhaustive.

Strateges 3 and 4 are usually randomized.

A generic heuristic search algorithm

Given N, a neighbourhood function, the heuristic algorithm h_N either:

- Perform one neighbourhood search (using one of the strategies)
- Perform a sequence of j neighbourhood searches, where each one takes us from X_i to X_{i+1}: $[X = X_0, X_1, \ldots, X_j = Y]$.

Algorithm $\textsc{GenericHeuristicSearch}(c_{max})$

Select a feasible solution $X \in \mathcal{X}$;
$X_{best} \leftarrow X$; (stores best so far); $c \leftarrow 0$;
while $(c \leq c_{max})$ do
  $Y \leftarrow h_N(X)$;
  if $(Y \neq$ "fail") then $X \leftarrow Y$;
    if $(P(X) > P(X_{best}))$ then $X_{best} \leftarrow X$;
  [else $c \leftarrow c_{max} + 1$; (add this if h_N is not randomized)]
  $c \leftarrow c + 1$;
return X_{best};

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  - $Y \in N(X)$ such that Y is feasible and $P(Y) > P(X)$,
  - or, otherwise, "fail".

Hill Climbing Algorithm

Algorithm $\text{GenericHillClimbing}()$

Select a feasible solution $X \in \mathcal{X}$.
$X_{best} \leftarrow X$; $searching \leftarrow true$;
while ($searching$) do
    $Y \leftarrow h_N(X)$;
    if ($Y \neq$ "fail") then
        $X \leftarrow Y$;
        if ($P(X) > P(X_{best})$) then $X_{best} \leftarrow X$;
    else $searching \leftarrow false$;
return X_{best};

Hill-climbing can get trapped in a local optimum.
Other search strategies (**simulated annealing**,**tabu search**) try to escape from local optima.

Simulated Annealing

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- Going uphill is always accepted.
- Going downhill is sometimes accepted with a probability based on how much downhill we go and on the current temperature.
  - Given $Y = h_N(X)$ with $P(Y) \leq P(X)$,
  - accept Y with probability

$$e^{(P(Y)-P(X))/T} = \frac{1}{e^{(P(X)-P(Y))/T}}$$

(We get pickier as we progress, since T decreases)

Simulated Annealing Algorithm

Algorithm GenericSimulatedAnnealing(c_{max}, T_0, α)
  $c \leftarrow 0$; $T \leftarrow T_0$;
  Select a feasible solution $X \in \mathcal{X}$; $X_{best} \leftarrow X$;
  while $(c \leq c_{max})$ do
    $Y \leftarrow h_N(X)$; // this is usually a randomized choice
    if $(Y \neq$ "fail"$)$ then
      if $(P(Y) > P(X))$ then
        $X \leftarrow Y$;
        if $(P(X) > P(X_{best}))$ then $X_{best} \leftarrow X$;
      else $r \leftarrow random(0, 1)$;
        if $(r < e^{\frac{P(Y)-P(X)}{T}})$ then $X \leftarrow Y$;
    $c \leftarrow c + 1$;
    $T \leftarrow \alpha T$;
  return X_{best};

Tabu Search

- Neighbourhood search:
  Choose $Y \in N(X) \setminus \{X\}$ such that Y is feasible and $P(Y)$ is maximum among all such elements (exhaustive neighbourhood search).
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  - Cycling may also take several steps, such as $X \to Y \to Z \to X$.

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- What may be the risk?
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  - When going downhill from X to Y we may go back from X to Y.
  - Cycling may also take several steps, such as $X \rightarrow Y \rightarrow Z \rightarrow X$.
- Tabu-search uses a strategy for avoiding cycling: a **tabu list**.
  After a move $X \rightarrow Y$,
  we forbit the application of $\textsc{change}(Y, X)$ for L iterations
  (L is the lifetime of the tabu list).

Tabu List

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- Example:
  $\mathcal{X} = \{0,1\}^n$, using $N_1(X) = \{Y \in \mathcal{X} : dist(X,Y) = 1\}$.
  $X = [0100]$ and $Y = [0101]$, we have that $\textsc{change}(Y, X) = 4 =$ index of coordinate that was swapped.
  Suppose $L = 2$.

sequence of points:	$[0100]$	$[0101]$	$[1101]$	$[1001]$	$[1011]$
tabu list:		4	4,1	1,2	2,3

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- So any sequence that cycles $X \rightarrow \ldots \rightarrow X$ has length at least $2L$. Choosing $L = 10$ is typical.

TabuList is defined below to be a list where TabuList[c] $= \delta$, where δ is the designated forbidden (tabu) change at iteration c.

For tabu search, $h_N(X) = Y$, where

- $Y \in N(X)$, Y is feasible;
- $\text{CHANGE}(X, Y) \notin \{ \text{TabuList}[d] : c - L \leq d \leq c - 1\}$;
- $P(Y)$ is maximum among all such feasible elements.

In absolute no circumstance implement TabuList as an array indexed by the number of iterations! Instead, implement TabuList as a queue of length L. Note that the algorithm may mislead you to think you are using such an array, given the notation defined above; careful!

Tabu Search Algorithm: textbook version/typo correction

Algorithm $\textsc{GenericTabuSearch}(c_{max}, L)$

$c \leftarrow 1$;
Select a feasible solution $X \in \mathcal{X}$.
$X_{best} \leftarrow X$;
while $(c \leq c_{max})$ do
  $N \leftarrow N(X) \setminus \{F : \textsc{change}(X, F) \in \textsc{TabuList}[d], c - L \leq d \leq c - 1\}$;
  for each $(Y \in N)$ do if (Y is infeasible) then $N \leftarrow N \setminus \{Y\}$;
  if $(N = \emptyset)$ then return X_{best};
  Find $Y \in N$ such that $P(Y)$ is maximum; /* computes $Y = h_N(X)$ */
  $\textsc{TabuList}[c] \leftarrow \textsc{change}(Y, X)$;
  $X \leftarrow Y$;
  if $(P(X) > P(X_{best}))$ then $X_{best} \leftarrow X$;
  $c \leftarrow c + 1$;
return X_{best};

Tabu List Implementation

In absolute no circumstance implement $\textsc{TabuList}$ as an array indexed by the number of iterations!

In the real implementation, $\textsc{TabuList}$ can be a queue of length L!!!

So, the operation

$\textsc{TabuList}[c] \leftarrow \textsc{change}(Y, X)$;

must be implemented as:

$\textsc{TabuList}$.insert($\textsc{change}(Y, X)$); (only keeps last L elements)

and the line: $N \leftarrow N(X) \setminus \{F : \textsc{change}(X, F) \in \textsc{TabuList}[d], c - L \leq d \leq c - 1\}$

should be understood as:

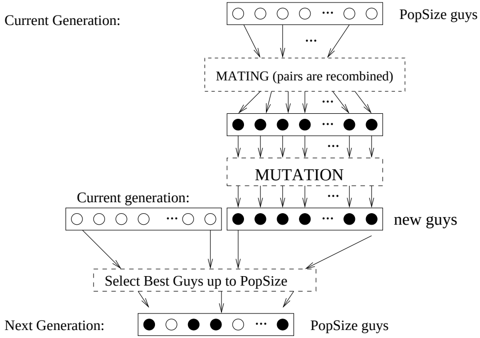
$N \leftarrow N(X) \setminus \{F : \textsc{change}(X, F)$ is in $\textsc{TabuList}\}$;

Tabu Search Algorithm: with FIFO queue for TABULIST

Algorithm GENERICTABUSEARCH(c_{max}, L)
  $c \leftarrow 1$;
  Select a feasible solution $X \in \mathcal{X}$.
  $X_{best} \leftarrow X$;
  while ($c \leq c_{max}$) do
    $N \leftarrow N(X) \setminus \{F :$ CHANGE(X, F) is in TABULIST$\}$
    for each ($Y \in N$) do   if (Y is infeasible) then $N \leftarrow N \setminus \{Y\}$;
    if ($N = \emptyset$) then return X_{best};
    Find $Y \in N$ such that $P(Y)$ is maximum; /* computes $Y = h_N(X)$ */
    TABULIST.insert(CHANGE$(Y, X), L$); /* only keeps last L entries */
    $X \leftarrow Y$;
    if ($P(X) > P(X_{best})$) then $X_{best} \leftarrow X$;
    $c \leftarrow c + 1$;
  return X_{best};

Genetic Algorithms

Fix a number PopSize (population size).
One iteration works as follows:



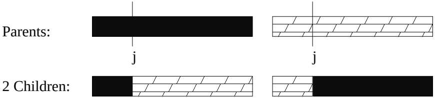
Iterate as many generations as you like.

Mating Strategies (Recombination)

Producing children from parents.

Method 1: Crossover.

Let j be a crossover point.



Example: $j = 3$

Parents:	$[110 \mid 1101001]$	$[100 \mid 1000101]$
Children:	$[110 \mid 1000101]$	$[100 \mid 1101001]$

Mating Strategies (Recombination), cont'd

Method 2: Partially matched crossover (for permutations)

Two crossover points: $1 \leq j < k \leq n$
Example: $j = 3$ and $k = 6$
$\alpha = [3, 1, \underline{4, 7, 6, 5}, 2, 8]$ $\beta = [8, 6, \underline{4, 3, 7, 1}, 2, 5]$

swap	α	β
$4 \leftrightarrow 4$	$[3, 1, 4, 7, 6, 5, 2, 8]$	$[8, 6, 4, 3, 7, 1, 2, 5]$
$7 \leftrightarrow 3$	$[7, 1, 4, 3, 6, 5, 2, 8]$	$[8, 6, 4, 7, 3, 1, 2, 5]$
$6 \leftrightarrow 7$	$[6, 1, 4, 3, 7, 5, 2, 8]$	$[8, 7, 4, 6, 3, 1, 2, 5]$
$5 \leftrightarrow 1$	$[6, 5, 4, 3, 7, 1, 2, 8]$	$[8, 7, 4, 6, 3, 5, 2, 1]$

Mating Schemes

Kids may be infeasible: incorporate constraints as penalties.

Various methods are possible for mating schemes:

1. Random monogamy with 2 kids per couple: randomly partition population into pairs, with two kids produced by each pair.
2. Make better parents having more kids:
   measure parent fitness by objective function; parents with higher fitness produce more kids.

Algorithm $\textsc{GenericGeneticAlgorithm}(PopSize, c_{max})$

Select an initial population $\mathcal{P}$ with $PopSize$ feasible solutions;
for each $X \in \mathcal{P}$ do $X \leftarrow h_N(X)$;   [mutation]
$X_{best} \leftarrow$ element in $\mathcal{P}$ with maximum profit; $c \leftarrow 1$;
while $(c \leq c_{max})$ do
    $\mathcal{Q} \leftarrow \mathcal{P}$; Construct a pairing of the elements in $\mathcal{P}$;
    for each pair (W, X) in the pairing do
        $(Y, Z) \leftarrow rec(W, X)$;   [recombination/mating]
        $Y \leftarrow h_N(Y)$; $Z \leftarrow h_N(Z)$;   [mutations]
        $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{Y, Z\}$;
    Set $\mathcal{P}$ to be the best $PopSize$ members of $\mathcal{Q}$;
    Let Y be the element in $\mathcal{P}$ with maximum profit;
    if $(P(Y) > P(X_{best}))$ then $X_{best} \leftarrow Y$;
    $c \leftarrow c + 1$;
return X_{best};

Steepest Ascent for Uniform Graph Partition

PROBLEM: UNIFORM GRAPH PARTITION
INSTANCE: A COMPLETE GRAPH ON $2n$ VERTICES,
$cost : E \to Z^+ \cup \{0\}$ (COST FUNCTION)
FIND: THE MINIMUM VALUE OF
$C([X_0, X_1]) = \sum_{u \in X_0, v \in X_1} cost(u, v)$
SUBJECT TO $V = X_0 \cup X_1,\ |X_0| = |X_1| = n$.

Example: $n = 4$; $cost(1,2) = 1, cost(1,3) = 2, cost(1,4) = 5, cost(2,3) = 0, cost(2,4) = 5, cost(3,4) = 1$.
Only 3 feasible solutions (except for exchanging X_0 and X_1):

$$X_0 = \{1,2\},\quad X_1 = \{3,4\},\qquad C([X_0, X_1]) = 12$$
$$X_0 = \{1,3\},\quad X_1 = \{2,4\},\qquad C([X_0, X_1]) = 7 \quad (optimal)$$
$$X_0 = \{1,4\},\quad X_1 = \{2,3\},\qquad C([X_0, X_1]) = 9$$

Uniform Graph Partition: Steepest Ascend Algorithm

Neighbourhood function: exchange $x \in X_0$ and $y \in X_1$.

Algorithm UGP(C_{max})
    $X = [X_0, X_1] \leftarrow$ SelectRandomPartition
    $c \leftarrow 1$
    while ($c \leq C_{max}$) do
        $[Y_0, Y_1] \leftarrow$ Ascend(X)
        if not fail then
                $\{X_0 \leftarrow Y_0;\ \ X_1 \leftarrow Y_1;\}$
        else return
        $c \leftarrow c + 1$

Ascend Algorithm

Algorithm Ascend($[X_0, X_1]$)
    $g \leftarrow 0$
    for each $i \in X_0$ do
        for each $j \in X_1$ do
            $t \leftarrow G_{[X_0,X_1]}(i,j)$ (gain obtained in exchange)
            if $(t > g)$ then $\{x \leftarrow i;\ y \leftarrow j;\ g \leftarrow t\}$
    if $(g > 0)$ then
        $Y_0 \leftarrow (X_0 \cup \{y\}) \setminus \{x\}$
        $Y_0 \leftarrow (X_1 \cup \{x\}) \setminus \{y\}$
        fail $\leftarrow$ false
        return $[Y_0, Y_1]$
    else $\{$fail $\leftarrow$ true; return $[X_0, X_1]\}$

SelectRandomPartition

Two possible algorithms:

1. Picking X_0 as a random n-subset r of a $2n$-set:
   Get a random integer $r \in [0, \binom{2n}{n} - 1]$ and apply
   $\texttt{kSubsetLexUnrank}(r, n, 2n)$.
2. Randomly shufling elements in $[0, 2n-1]$:
   Create array $A[0, 2n-1]$ with randomly chosen numbers as elements.
   Create array $B[0, 2n-1]$ initially with $B[i] = i$.
   Sort A, doing same swaps on B.
   Take X_0 as the first half of B, and X_1 as the second half.

Hill-climbing for Steiner triple systems

Textbook, Section 5.4.

Definition

A *Steiner triple system* of order v, denoted $STS(v)$, is a pair $(V, \mathcal{B})$ where:
$V = \{1, 2, \ldots v\}$ is a set of points,
$\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is a set of 3-sets, called blocks, such that any pair of points in V is in a unique block $B_i \in \mathcal{B}$.

Example: $STS(9)$:

$$\begin{aligned} V = & \quad \{1,2,3,4,5,6,7,8,9\} \\ \mathcal{B} = \{ & \quad \{1,2,3\},\{1,4,7\},\{1,5,9\},\{1,6,8\},\{4,5,6\},\{2,5,8\}, \\ & \quad \{2,6,7\},\{2,4,9\},\{7,8,9\},\{3,6,9\},\{3,4,8\},\{3,5,7\}\} \end{aligned}$$

Replication number and number of blocks

Lemma

Let $(V, \mathcal{B})$ be an $STS(v)$. Then, every point in V occurs in exactly $r = \frac{v-1}{2}$ blocks and $|\mathcal{B}| = \frac{v(v-1)}{6}$.

PROOF:

1. Any point x must appear in some block with each of all other $(v-1)$ points. Point x occurs with 2 other points in each of the r_x blocks it appears. Therefore, $r_x = \frac{v-1}{2}$.
2. We count T, the number of points with their replications appearing on $\mathcal{B}$, in two ways: $T = 3 \times b$ and $T = v \times r$. Thus, $3 \times b = v \times r$, which implies $b = \frac{v(v-1)}{6}$.

Necessary and sufficient conditions for existence of $STS(v)$

Since $r = \frac{v-1}{2}$ (point replication number) and $b = \frac{v(v-1)}{6}$ (number of blocks) must be integer numbers, we need $v \equiv 1, 3 \pmod 6$.
These necessary conditions have been proven to be sufficient:

Theorem

$$\exists STS(v) \iff v \equiv 1, 3 \pmod 6$$

So, there exists an $STS(v)$ for
$v = 1, 3, 7, 9, 13, 15, 19, 21, 25, 17, 31, 33, \ldots$

A partial Steiner triple system consists of a set of triples $\mathcal{B}$ with each pair of points appearing in at most one $B_i \in \mathcal{B}$. Then, we can formulate the search problem as follows.

Searching for Steiner Triple Systems

PROBLEM:	CONSTRUCT A STEINER TRIPLE SYSTEM
INSTANCE:	v SUCH THAT $v \equiv 1, 3 \pmod 6$
FIND:	MAXIMIZE $\lvert\mathcal{B}\rvert$
	SUBJECT TO: $([1, v], \mathcal{B})$ IS A PARTIAL STEINER TRIPLE SYSTEM

The **universe** $\mathcal{X}$ is the set of all sets of blocks $\mathcal{B}$, such that $([1, v], \mathcal{B})$ is a partial Steiner triple system.

An **optimal solution** is any feasible solution with $|\mathcal{B}| = \frac{v(v-1)}{6}$.

Stinson's hill-climbing algorithm for STSs

Algorithm Stinson's Algorithm(v)
  Numblocks $\leftarrow 0$
  $V \leftarrow \{1, 2, \ldots v\}$
  $\mathcal{B} \leftarrow \emptyset$
  While (Numblocks $< \frac{v(v-1)}{2}$) do { SWITCH}
  output $(V, \mathcal{B})$

To present SWITCH, we need:

Definition

A point x is said to be a **live point** in $([1, v], \mathcal{B})$ if $r_x < \frac{v-1}{2}$.
A pair $\{x, y\}$ is said to be a **live pair** in $([1, v], \mathcal{B})$ if there exists no $B \in \mathcal{B}$ with $\{x, y\} \subseteq B$

Stinson's hill-climbing for STSs: Switch Algorithm

```
Algorithm SWITCH
    Chosse a random live point x.
    Choose random y, z such that
      {x, y} and {x, z} are live pairs.
    If ({y, z} is a live pair) then
       B ← B ∪ {{x, y, z}}
       Numblocks ← Numblocks +1
    else
        Let {w, y, z} ∈ B be the block containing {y, z}
        B ← B ∪ {{x, y, z}} \ {{w, y, z}}
```

See implementation details in the textbook.

Using appropriatte data structures, SWITCH is implemented in constant time.

Two heuristics for the Knapsack Problem

Knapsack (Optimization) Problem

Instance: Profits $p_0, p_1, \ldots, p_{n-1}$
Weights $w_0, w_1, \ldots, w_{n-1}$
Knapsack capacity M

Universe: $\mathcal{X} = \{0, 1\}^n$ (set of all n-tuples)
an n-tuple $[x_0, x_1, \ldots, x_{n-1}]$ is feasible if
$\sum_{i=0}^{n-1} w_i x_i \leq M$.

Objective: maximize $P(X) = \sum_{i=0}^{n-1} p_i x_i$.

Algorithm $\text{KNAPSACKSIMULATEDANNEALING}(c_{max}, T_0, \alpha)$

$c \leftarrow 0$; $T \leftarrow T_0$; $X \leftarrow [x_0, x_1, \ldots, x_{n-1}] = [0, 0, \ldots, 0]$;
$CurW \leftarrow 0$; $X_{best} \leftarrow X$;
while $(c \leq c_{max})$ do
  $j \leftarrow$ randomInt$(0, n-1)$; $Y \leftarrow X$; $y_j \leftarrow 1 - x_j$; (using $N_1(X)$)
  if $(y_j = 1)$ and $(curW + w_j > M)$ then $Y \leftarrow fail$;
  if $(Y \neq fail)$ then if $(y_j = 1)$ then
    $X \leftarrow Y$;
    $curW \leftarrow curW + w_j$;
    if $P(X) > P(X_{best})$ then $X_{best} \leftarrow X$;
  else $r \leftarrow$ random$(0, 1)$;
    if $(r < e^{-p_j/T})$ then
      $X \leftarrow Y$; $curW \leftarrow curW - w_j$;
  $c \leftarrow c + 1$; $T \leftarrow \alpha T$;
return (X_{best});

Knapsack Simulated Annealing Results

TABLE 5.3
Summary data for the knapsack simulated annealing algorithm.

α	c_{max}	profits found		
		minimum	maximum	average
0.999	1000	1441	1454	1446.8
0.999	5000	1448	1456	1452.1
0.999	20000	1448	1456	1450.9
0.9995	1000	1445	1455	1448.4
0.9995	5000	1450	1458	1454.6
0.9995	20000	1452	1458	1453.9
0.9999	1000	1445	1455	1449.6
0.9999	5000	1450	1458	1454.3
0.9999	20000	1453	1458	1456.1

Tabu Search for Knapsack

We will use the same neighbourhood $N_1(.)$.

Do exhaustive search on the neighbourhood in order to find the best way to update the current solution.

Instead of Profit improvement only, we look for improvements based on the ratio p_i/w_i:

1. Chose i with maximum p_i/w_i among the indexes j where $x_j = 0$, j is not on $\textsc{TabuList}$, and changing x_j to 1 does not exceed M.
2. If there is no j as above, then choose i with minimum p_i/w_i among the indexes j where $x_j = 1$ and j is not on $\textsc{TabuList}$.

This can be expressed by saying that we want to maximize

$$(-1)^{x_j} \frac{p_j}{w_j}.$$

Algorithm $\textsc{KnapsackTabuSearch}(c_{max}, L)$

Select a random feasible $X = [x_0, x_1, \ldots, x_{n-1}] \in \{0,1\}^n$;
$curW \leftarrow \sum_{i=0}^{n-1} x_i w_i$; $X_{best} \leftarrow X$;
for $(c \leftarrow 1; c \leq c_{max}; c \leftarrow c+1)$ do
    $N \leftarrow \{0, 1, \ldots, n-1\} \setminus \{j : j$ is in $\textsc{TabuList}\}$;
    for each $(i \in N)$ do
        if $(x_i = 0)$ and $(curW + w_i > M)$ then $N \leftarrow N \setminus \{i\}$;
    if $(N = \emptyset)$ then break for-loop;
    Find $i \in N$ such that $(-1)^{x_i} p_i / w_i$ is maximum;
    $\textsc{TabuList.insert}(i, L)$; (removing oldest, if has $L+1$ items)
    $x_i \leftarrow 1 - x_i$;   (swap i coordinate)
    if $(x_i = 1)$ then $curW \leftarrow curW + w_i$;
        else $curW \leftarrow curW - w_i$;
    if $P(X) > P(X_{best})$ then $X_{best} \leftarrow X$;
return X_{best};

TABLE 5.6
Summary data for the knapsack tabu search algorithm (24 items).

L	profits found (in 25 runs)			# optimal solutions found
	minimum	maximum	average	
1	13079298	13466838	13388643.5	0
2	13084476	13500943	13415747.5	0
3	13245597	13500943	13456205.2	0
4	13264009	13500943	13446933.8	0
5	13358351	13500943	13458145.8	0
6	13148978	13549094	13427333.6	1
7	13116665	13549094	13462902.4	4
8	13346220	13549094	13497932.2	7

A Genetic Algorithm for the TSP

Traveling Salesman Problem (TSP)

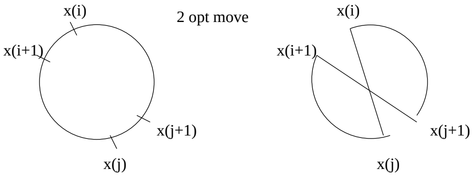
Instance: a complete graph K_n
a cost function $c : V \times V \to R$
Find: a Hamiltonian circuit $[x_0, x_1, \ldots, x_{n-1}]$ that minimizes
$C(X) = c(x_0, x_1) + c(x_1, x_2) + \ldots + c(x_{n-1}, x_0)$

Note that $2n$ permutations represent the same cycle.
Universe: $\mathcal{X}$ = set of all $n!$ permutations.
Steps:

- Selection of initial population.
- Mutation: steepest ascent 2-opt.
- Recombination using two methods: partially matched crossover and another method.

Mutation

Steepest ascent algorithm based on the 2-opt heuristic:



Gain in applying a 2-opt move:

$$\begin{aligned} G(X,i,j) &= C(X) - C(X_{ij}) \\ &= c(x_i, x_{i+1}) + c(x_j, x_{j+1}) - c(x_{i+1}, x_{j+1}) - c(x_i, x_j) \end{aligned}$$

$N(X) =$ all $Y \in \mathcal{X}$ that can be obtained from X by a 2-opt move.

Algorithm $\textsc{SteepestAscentTwoOpt}(X)$

$done \leftarrow false$;
while $(not\ done)$ do
  $done \leftarrow true$; $g_0 \leftarrow 0$;
  for $i \leftarrow 0$ to $n-1$ do
    for $j \leftarrow i+2$ to $n-1$ do
      $g \leftarrow G(X, i, j)$;
      if $(g > g_0)$ then
        $g_0 \leftarrow g$; $i_0 \leftarrow i$; $j_0 \leftarrow j$;
  if $(g_0 > 0)$ then
    $X \leftarrow X_{i_0, j_0}$;
    $done \leftarrow false$;

Selecting the initial population

Randomly pick one and then mutate it:

Algorithm $\textsc{Select}(popsize)$
for $i \leftarrow 0$ to $popsize - 1$ do
  $r \leftarrow \textsc{RandomInteger}(0, n! - 1)$;
  $P_i \leftarrow \textsc{PermLexUnrank}(n, r)$;
  $\textsc{SteepestAscentTwoOpt}(P_i)$;
return $[P_0, P_1, \ldots, P_{popsize-1}]$;

Recombination algorithm 1: Partially Matched Crossover

Algorithm $\text{PMR}\textsc{ec}(A, B)$
$h \leftarrow \textsc{RandomInteger}(10, n/2)$; (length of the substring)
$j \leftarrow \textsc{RandomInteger}(0, n-1)$; (start of the substring)
$(C, D) \leftarrow \textsc{PartiallyMatchedCrossover}(A, B, j, (h+j) mod\ n)$
$\textsc{SteepestAscentTwoOpt}(C)$;
$\textsc{SteepestAscentTwoOpt}(D)$;
return (C, D);

Recombination Algorithm 2

Algorithm $\text{MGKRec}(A, B)$

$h \leftarrow \text{RandomInteger}(10, n/2)$; (length of the substring)
$j \leftarrow \text{RandomInteger}(0, n-1)$; (start of the substring)
$T \leftarrow \emptyset$;
(pick subcycle of length h starting from pos j:)
for $i \leftarrow 0$ to $h-1$ do
  $D[i] \leftarrow B[(i+j) \mod n]$;
  $T \leftarrow T \cup \{D[i]\}$;
Complete cycle with permutation in A using guys not already in D in the order prescribed by A:
for $j \leftarrow 0$ to $n-1$ do
  if $A[j] \notin T$ then $\{D[i] \leftarrow A[j]; \ i \leftarrow i+1; \}$
$\text{SteepestAscentTwoOpt}(D)$;
(Similarly build C swapping A and B roles:)...

(Algorithm continued)

(Similarly build C swapping A and B roles:)...
$j \leftarrow \text{RandomInteger}(0, n-1)$; (start of the substring)
$T \leftarrow \emptyset$;
for $i \leftarrow 0$ to $h-1$ do
    $C[i] \leftarrow A[(i+j) \mod n]$;
    $T \leftarrow T \cup \{C[i]\}$;
for $j \leftarrow 0$ to $n-1$ do
    if $B[j] \notin T$ then $\{C[i] \leftarrow B[j];\ i \leftarrow i+1;\ \}$
$\text{SteepestAscentTwoOpt}(C)$;
return (C, D);

Genetic Algorithm for TSP

Algorithm $\textsc{GeneticTSP}(popsize, c_{max})$

$[P_0, P_1, \ldots, P_{popsize-1}] \leftarrow \textsc{Select}(popsize)$;
Sort $P_0, P_1, \ldots, P_{popsize-1}$ in increasing order of cost.
$X_{best} \leftarrow P_0$; $BestCost \leftarrow C(P_0)$;
for $(c \leftarrow 1; c \leq c_{max}; c \leftarrow c+1)$ do
    for $i \leftarrow 0$ to $popsize/2 - 1$ do
        $(P_{popsize+2i}, P_{popsize+2i+1}) \leftarrow \textsc{Rec}\ (P_{2i}, P_{2i+1})$;
    Sort $P_0, P_1, \ldots, P_{2popsize-1}$ in increasing order of cost.
    $curCost \leftarrow C(P_0)$;
    if $(curCost < BestCost)$ then
        $X_{best} \leftarrow P_0$;
        $BestCost \leftarrow curCost$;
return X_{best};

TABLE 5.7
GENETICTSP data with recombination operation PMREC.

M	n	Opt. Cost	*popsize*	c_{max}	cost found min	max	avg	No. Opt. found
M50a	50	185	8	50	192	214	200.50	0
				100	191	219	200.00	0
				200	190	203	196.60	0
			16	50	187	207	193.20	0
				100	187	206	193.20	0
				200	187	200	193.70	0
			32	50	189	205	194.70	0
				100	186	199	190.70	0
				200	188	200	192.40	0
M50b	50	158	8	50	163	184	175.40	0
				100	163	195	173.70	0
				200	160	191	177.30	0
			16	50	159	176	167.40	0
				100	163	184	171.50	0
				200	161	189	172.10	0
			32	50	161	173	167.60	0
				100	163	178	169.40	0
				200	159	178	166.70	0
M50c	50	155	8	50	162	181	169.40	0
				100	159	186	169.50	0
				200	159	187	169.30	0
			16	50	155	171	161.30	1
				100	155	182	166.10	1
				200	157	182	167.70	0
			32	50	155	170	161.60	1
				100	158	167	161.40	0
				200	157	180	162.50	0

A Genetic Algorithm for TSP

TABLE 5.8
GENETICTSP with recombination operation MGKREC.

M	n	Opt. Cost	*popsize*	c_{max}	cost found min	max	avg	No. Opt. found
M50a	50	185	8	50	186	196	191.70	0
				100	186	199	190.30	0
				200	186	194	189.20	0
			16	50	186	192	189.20	0
				100	185	192	187.00	3
				200	185	192	187.60	1
			32	50	186	192	188.10	0
				100	185	190	187.30	1
				200	185	190	187.30	1
M50b	50	158	8	50	160	171	165.30	0
				100	159	166	161.60	0
				200	159	170	162.00	0
			16	50	158	164	161.20	1
				100	158	162	159.80	1
				200	159	163	160.70	0
			32	50	158	165	160.70	1
				100	159	163	160.30	0
				200	158	160	158.90	2
M50c	50	155	8	50	156	168	160.50	0
				100	155	167	160.70	2
				200	155	162	157.30	5
			16	50	155	162	157.50	2
				100	155	159	156.30	5
				200	155	159	155.70	8
			32	50	155	159	156.10	6
				100	155	158	155.40	8
				200	155	156	155.10	9