HEURISTIC SEARCH

Heuristic Search vs Exhaustive Search

Exhaustive Search

Types of methods and their uses:

- Backtracking (backtracking with bounding):
  - Find all feasible solutions.
  - Find one optimal solution.
  - Find all optimal solutions.
- Branch-and-Bound:
  - Find one optimal solution.

Heuristic Search

Types of problem it can be applied to:

- Find 1 optimal solution.
- Find a "close to" optimal solution (the best solution we manage).

Heuristics methods we will study:

- Hill-climbing
- Simulated annealing
- Tabu search
- Genetic algorithm

Characteristics of heuristic search:

- The state space is not fully explored.
- Randomization is often employed.
- There is a concept of neighbourhood search.
- **Heuristics** are applied to explore the solutions.
  The word "heuristics" means "serving or helping to find or discover" or "proceeding by trial and error".

A general framework for heuristic search

Generic Optimization Problem (maximization):

Instance: A finite set $\mathcal{X}$.
an objective function $P : \mathcal{X} \rightarrow Z$.
m feasibility functions $g_j : \mathcal{X} \rightarrow Z$, $1 \leq j \leq m$.

Find: the maximum value of $P(X)$
subject to $X \in \mathcal{X}$ and $g_j(X) \geq 0$, for $1 \leq j \leq m$.

Exercise: pick your favorite combinatorial optimization problem and write it in this framework.

Designing a heuristic search:

1. Define a **neighbourhood function** $N : \mathcal{X} \rightarrow 2^{\mathcal{X}}$.

   E.g. $N(X) = \{X_1, X_2, X_3, X_4, X_5\}$.

2. Design a **neighbourhood search**:
   Algorithm that finds a fasible solution on the neighbourhood of a feasible solution X.

   There are two types of neghbourhood searches:
   - Exhaustive (chooses best profit among neighbour points)
   - Randomized (picks a random point among the neighbour points)

Defining a neighbourhood function

$N : \mathcal{X} \to 2^{\mathcal{X}}$, $N(X) \subseteq \mathcal{X}$.
$N(X)$ should be elements that are similar or "close to" X.
$N(X)$ may contain infesible elements of $\mathcal{X}$.

Examples of neighbourhood functions:
Let d_0 be a constant positive integer.

$$N_{d_0}(X) = \{Y \in \mathcal{X} : dist(X,Y) \leq d_0\},$$

- $\mathcal{X} = \{0,1\}^n$, all binary n-tuples.
  Here $dist$ is the Hamming distance.
  $N_1([010]) = \{[000],[110],[011],[010]\}$.

$$|N_{d_0}(X)| = \sum_{i=0}^{d_0} \binom{n}{i}.$$

- $\mathcal{X}$ = all permutations of $\{1,2,\ldots,n\}$.
  Let $\alpha = [\alpha_1,\ldots,\alpha_n]$ and $\beta = [\beta_1,\ldots,\beta_n]$ be two permutations.
  Define distance as follows: $dist(\alpha,\beta) = |\{i : \alpha_i \neq \beta_i\}|$.
  Note that $N_1(X) = \{X\}$ is not very useful; we need $d_0 > 1$.
  $N_2([1,2,3,4]) = \{[1,2,3,4],[2,1,3,4],[3,2,1,4],$
  $[4,2,3,1],[1,3,2,4],[1,4,3,2],[1,2,4,3]\}$

  $|N_2(X)| = 1 + \binom{n}{2}$.

Designing a neighbourhood search algorithm

Neighbourhood Search Algorithm

Input: X
Output: $Y \in N(X) \setminus \{X\}$ such that Y is feasible, or "fail".

Possible Neighbourhood Search Strategies:

1. Find a feasible solution $Y \in N(X) \setminus \{X\}$ such that $P(Y)$ is maximized.
   Return "fail" if there is no feasible solution in $N(X) \setminus \{X\}$.

2. Find a feasible solution $Y \in N(X) \setminus \{X\}$ such that $P(Y)$ is maximized.
   if $P(Y) > P(X)$ then return Y; else return "fail".
   (steepest ascent method)

3. Find any feasible solution $Y \in N(X) \setminus \{X\}$.
   Return "fail" if there is no feasible solution in $N(X) \setminus \{X\}$.

4. Find any feasible solution $Y \in N(X) \setminus \{X\}$.
   if $P(Y) > P(X)$ then return Y; else return "fail".

Strategies 1 and 2 may be exhaustive.
Strateges 3 and 4 are usually randomized.

A generic heuristic search algorithm

Given N, a neighbourhood function, the heuristic algorithm h_N does either of the following:

- Perform one neighbourhood search (using one of the strategies)
- Perform a sequence of j neighbourhood searches $[X = X_0, X_1, \ldots, X_j = Y]$, where you get from X_i to X_{i+1} through a neighbourhood search.

Let c_{max} be the maximum number of iterations allowed for the search.

Algorithm GenericHeuristicSearch(c_{max})

```
    c ← 0;
    Select a feasible solution X ∈ 𝒳;
    X_best ← X; (stores best so far)
    while (c ≤ c_max) do
        Y ← h_N(X);
        if (Y ≠ "fail") then
            X ← Y;
            if (P(X) > P(X_best)) then X_best ← X;
        [else c ← c_max + 1; (add this if h_N is not randomized)]
        c ← c + 1;
    return X_best;
```

Design Strategies for Heuristic Algorithms

1. Hill-Climbing

Idea: Go up the hill continuously, stop when stuck.
Problem: it can get stuck in a local optimum.
Improvement: run the algorithm many times from random start X.

For Hill-Climbing, $h_N(X)$ returns:

- $Y \in N(X)$ such that Y is feasible and $P(Y) > P(X)$,
- or, otherwise, "fail".

Algorithm GenericHillClimbing()

```
Select a feasible solution X ∈ 𝒳.
X_best ← X; searching ← true;
while (searching) do
    Y ← h_N(X);
    if (Y ≠ "fail") then
      X ← Y;
      if (P(X) > P(X_best)) then X_best ← X;
    else searching ← false;
return X_best;
```

Hill-climbing will get trapped in a local optimum.
Other search strategies, such as simulated annealing and tabu search, try to escape from local optima.

2. Simulated annealing

- Analogy with a method of cooling metal: annealing. Temperature T decreases at each iteration, according to a **cooling schedule**: Initally $T \leftarrow T_0$; later $T \leftarrow \alpha T$ for a fixed $0 < \alpha < 1$.
- Going uphill is always accepted.
- Going downhill is sometimes accepted with a probability based on how much downhill we go and on the current temperature. Given $Y = h_N(X)$ with $P(Y) \leq P(X)$, accept Y with probability $e^{\frac{P(Y)-P(X)}{T}}$. (We get pickier as we progress.)

```
Algorithm GenericSimulatedAnnealing(c_max, T_0, α)
    c ← 0; T ← T_0;
    Select a feasible solution X ∈ 𝒳; X_best ← X;
    while (c ≤ c_max) do
        Y ← h_N(X); // this is usually a randomized choice
        if (Y ≠ "fail") then
          if (P(Y) > P(X)) then
            X ← Y;
            if (P(X) > P(X_best)) then X_best ← X;
          else
            r ← random(0, 1);
            if (r < e^((P(Y)−P(X))/T)) then X ← Y;
        c ← c + 1;
        T ← αT;
    return X_best;
```

3. Tabu Search

Choose $Y \in N(X) \setminus \{X\}$ such that Y is feasible and $P(Y)$ is maximum among all such elements (exhaustive neighbourhood search).
It may happen that $P(Y) < P(X)$ (we escape from a local optimum).

What may be the risk? Cycling.
When going downhill from X to Y we may go back from X to Y.
Indeed, cycling may take several steps, such as
$X \to Y \to Z \to X$.

Tabu-search uses a strategy for avoiding cycling: a **tabu list**.
After a move $X \to Y$,
we forbit the application of CHANGE(Y, X) for L iterations (L is the lifetime of the tabu list).

Example:
$\mathcal{X} = \{0,1\}^n$, using $N_1(X) = \{Y \in \mathcal{X} : dist(X,Y) = 1\}$.
$X = [0100]$ and $Y = [0101]$, we have that CHANGE$(Y, X) = 4 =$ index of coordinate that was swapped.

Suppose $L = 2$.

sequence of points:	[0100]	[0101]	[1101]	[1001]	[1011]
tabu list:		4	4,1	1,2	2,3

So any sequence that cycles $X \rightarrow \ldots \rightarrow X$ has length at least $2L$. Choosing $L = 10$ is typical.

TabuList is implemented as a list where TabuList[c] $= \delta$, where δ is the designated forbidden (tabu) change at iteration c.

In absolute no circumstance implement TabuList as an array indexed by the number of iterations! Instead, implement TabuList as a queue of length L. Note that the algorithm may mislead you to think you are using such an array; careful!

For tabu search, $h_N(X) = Y$, where

- $Y \in N(X)$, Y is feasible;
- change$(X, Y) \notin \{$ TabuList$[d] : c - L \leq d \leq c - 1\}$;
- $P(Y)$ is maximum among all such feasible elements.

Algorithm GENERICTABUSEARCH(c_{max}, L)

$c \leftarrow 1$;
Select a feasible solution $X \in \mathcal{X}$.
$X_{best} \leftarrow X$;
while ($c \leq c_{max}$) do
  $N \leftarrow N(X) \setminus \{F : \text{CHANGE}(X, F) \in \text{TABULIST}[d],$
    $c - L \leq d \leq c - 1\}$; (typo corrected)
  for each ($Y \in N$) do
    if (Y is infeasible) then $N \leftarrow N \setminus \{Y\}$;
  if ($N = \emptyset$) then return X_{best};
  Find $Y \in N$ such that $P(Y)$ is maximum;
  TABULIST[c] $\leftarrow$CHANGE(Y, X);
  $X \leftarrow Y$;
  if ($P(X) > P(X_{best})$) then $X_{best} \leftarrow X$;
  $c \leftarrow c + 1$;
return X_{best};

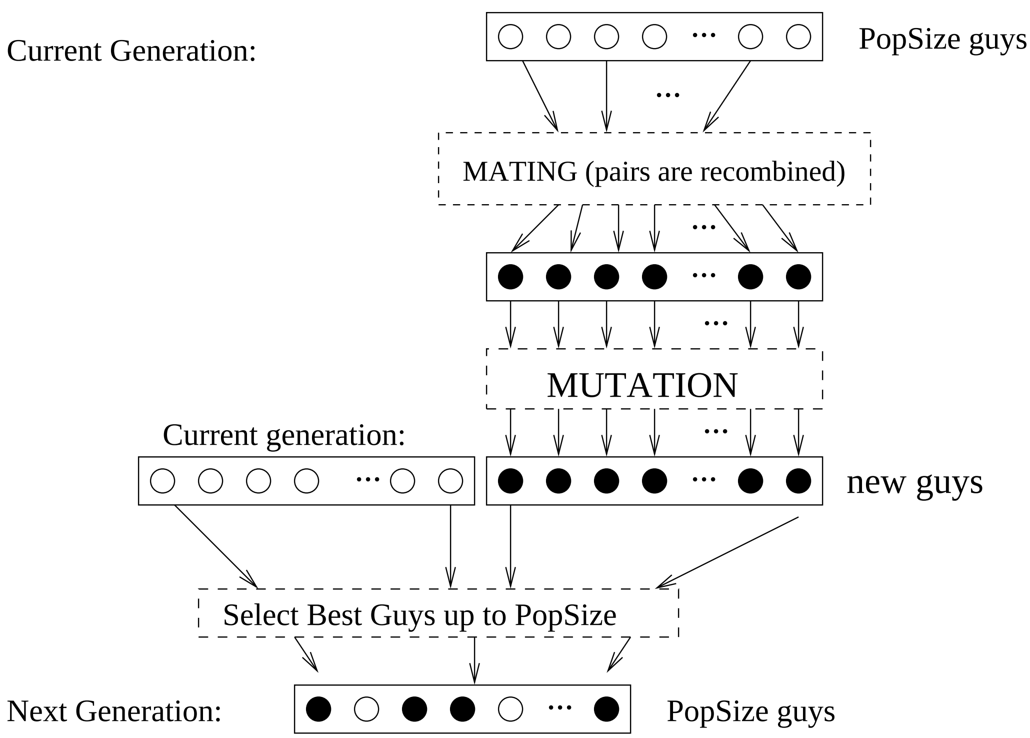
In the real algorithm, TABULIST must be a queue of length L!!!

So, the operation
TABULIST[c] $\leftarrow$CHANGE(Y, X);
must be implemented as:
TABULIST.insert(CHANGE(Y, X)); (only keeps last L elements)
and the line: $N \leftarrow N(X) \setminus \{F :$
CHANGE(X, F) $\in$TABULIST[d], $c - L \leq d \leq c - 1\}$
should be understood as:
$N \leftarrow N(X) \setminus \{F : \text{CHANGE}(X, F) \in \text{TABULIST}\}$;

4. Genetic Algorithms

More complex than neighbourhood search.
Fix a number PopSize (population size).

One iteration works as follows:

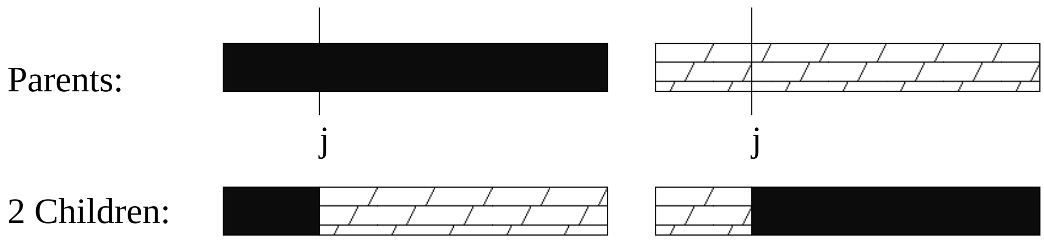


Iterate as many generations as you like.

Mating Strategies (Recombination)

Producing children from parents.

1. Crossover.
   Let j be a crossover point.



   Example: $j = 3$

Parents:	[110|1101001]	[100|1000101]
Children:	[110|1000101]	[100|1101001]

2. Partially matched crossover (for permutations)

   Two crossover points: $1 \leq j < k \leq n$

   Example: $j = 3$ and $k = 6$
   $\alpha = [3, 1, \underline{4, 7, 6, 5}, 2, 8]$ $\quad \beta = [8, 6, \underline{4, 3, 7, 1}, 2, 5]$

swap	α	β
$4 \leftrightarrow 4$	$[3, 1, 4, 7, 6, 5, 2, 8]$	$[8, 6, 4, 3, 7, 1, 2, 5]$
$7 \leftrightarrow 3$	$[7, 1, 4, 3, 6, 5, 2, 8]$	$[8, 6, 4, 7, 3, 1, 2, 5]$
$6 \leftrightarrow 7$	$[6, 1, 4, 3, 7, 5, 2, 8]$	$[8, 7, 4, 6, 3, 1, 2, 5]$
$5 \leftrightarrow 1$	$[6, 5, 4, 3, 7, 1, 2, 8]$	$[8, 7, 4, 6, 3, 5, 2, 1]$

Mating Schemes

Kids may be infeasible: incorporate constraints as penalties.

- Random monogamy with 2 kids per couple: randomly partition population into pairs, with two kids produced by each pair.
- Make better parents having more kids:
  measure parent fitness by objective function; parents with higher fitness produce more kids.

Algorithm GENERICGENETICALGORITHM($PopSize, c_{max}$)

```
c ← 1;
Select an initial population 𝒫 with PopSize feasible solutions;
for each X ∈ 𝒫 do X ← h_N(X);    [mutation]
X_best ← element in 𝒫 with maximum profit;
while (c ≤ c_max) do
    Construct a pairing of the elements in 𝒫;
    𝒬 ← 𝒫
    for each pair (W, X) in the pairing do
        (Y, Z) ← rec(W, X);    [recombination/mating]
        Y ← h_N(Y);    [mutation]
        Z ← h_N(Z);    [mutation]
        𝒬 ← 𝒬 ∪ {Y, Z};
    Let 𝒫 be the best PopSize members of 𝒬;
    Let Y be the element in 𝒫 with maximum profit;
    if (P(Y) > P(X_best)) then X_best ← Y;
    c ← c + 1;
return X_best;
```

HEURISTIC SEARCHES APPLIED TO VARIOUS PROBLEMS

Hill-climbing Algorithms

Steepest Ascent for Uniform Graph Partition

Textbook, Section 5.3.

PROBLEM: UNIFORM GRAPH PARTITION
INSTANCE: A COMPLETE GRAPH ON $2n$ VERTICES,
$cost : E \rightarrow Z^+ \cup \{0\}$ (COST FUNCTION)
FIND: THE MINIMUM VALUE OF
$C([X_0, X_1]) = \Sigma_{u \in X_0, v \in X_1} cost(u, v)$
SUBJECT TO $V = X_0 \cup X_1,\ |X_0| = |X_1| = n.$

Example: $n = 4$
$cost(1, 2) = 1, cost(1, 3) = 2, cost(1, 4) = 5,$
$cost(2, 3) = 0, cost(2, 4) = 5, cost(3, 4) = 1.$

Only 3 feasible solutions (except for exchanging X_0 and X_1):

$X_0 = \{1, 2\},\quad X_1 = \{3, 4\},\qquad C([X_0, X_1]) = 12$
$X_0 = \{1, 3\},\quad X_1 = \{2, 4\},\qquad C([X_0, X_1]) = 7 \quad (optimal)$
$X_0 = \{1, 4\},\quad X_1 = \{2, 3\},\qquad C([X_0, X_1]) = 9$

Neighbourhood function: exchange $x \in X_0$ and $y \in X_1$.

```
Algorithm UGP(C_max)
    X = [X_0, X_1] ← SelectRandomPartition
    c ← 1
    while (c ≤ C_max) do
        [Y_0, Y_1] ← Ascend(X)
        if not fail then
                {X_0 ← Y_0;  X_1 ← Y_1;}
        else return
        c ← c + 1


Algorithm Ascend([X_0, X_1])
    g ← 0
    for each i ∈ X_0 do
        for each j ∈ X_1 do
            t ← G_[X_0,X_1](i, j)   (gain obtained in exchange)
            if (t > g) then {x ← i;  y ← j;  g ← t}
    if (g > 0) then
        Y_0 ← (X_0 ∪ {y}) \ {x}
        Y_0 ← (X_1 ∪ {x}) \ {y}
        fail ← false
        return [Y_0, Y_1]
    else {fail ← true; return [X_0, X_1]}
```

Two possible algorithms for **`SelectRandomPartition`**:

- Picking X_0 as a random n-subset r of a $2n$-set:
  Get a random integer $r \in [0, \binom{2n}{n} - 1]$ and apply
  `kSubsetLexUnrank`$(r, n, 2n)$.
- Randomly shufling elements in $[0, 2n - 1]$:
  Create array $A[0, 2n - 1]$ with randomly chosen numbers as elements.
  Create array $B[0, 2n - 1]$ initially with $B[i] = i$.
  Sort A, doing same swaps on B.
  Take X_0 as the first half of B, and X_1 as the second half.

Hill-climbing for Steiner triple systems

Textbook, Section 5.4.

DEFINITION. A *Steiner triple system* of order v, denoted $STS(v)$, is a pair $(V, \mathcal{B})$ where:
$V = \{1, 2, \ldots v\}$ is a set of points,
$\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is a set of 3-sets, called blocks, such that any pair of points in V is in a unique block $B_i \in \mathcal{B}$.

Example: $STS(9)$:

$$\begin{array}{rl} V = & \{1,2,3,4,5,6,7,8,9\} \\ \mathcal{B} = \{ & \{1,2,3\},\{1,4,7\},\{1,5,9\},\{1,6,8\},\{4,5,6\},\{2,5,8\}, \\ & \{2,6,7\},\{2,4,9\},\{7,8,9\},\{3,6,9\},\{3,4,8\},\{3,5,7\}\} \end{array}$$

LEMMA. Let $(V, \mathcal{B})$ be an $STS(v)$. Then, every point in V occurs in exactly $r = \frac{v-1}{2}$ blocks and $|\mathcal{B}| = \frac{v(v-1)}{6}$.
PROOF:

1. Any point x must appear in some block with each of all other $(v-1)$ points. Point x occurs with 2 other points in each of the r_x blocks it appears. Therefore, $r_x = \frac{v-1}{2}$.
2. We count T, the number of points with their replications appearing on $\mathcal{B}$, in two ways: $T = 3 \times b$ and $T = v \times r$. Thus, $3 \times b = v \times r$, which implies $b = \frac{v(v-1)}{6}$.

Necessary conditions for existence of $STS(v)$:
Since $r = \frac{v-1}{2}$ (point replication number) and $b = \frac{v(v-1)}{6}$ (number of blocks) must be integer numbers, we need $v \equiv 1, 3 \pmod 6$.

The necessary conditions have been proven to be sufficient:

$$\exists STS(v) \iff v \equiv 1, 3 \pmod 6$$

So, there exists an $STS(v)$ for
$v = 1, 3, 7, 9, 13, 15, 19, 21, 25, 17, 31, 33, \ldots$

A partial Steiner triple system consists of a set of triples $\mathcal{B}$ with each pair of points appearing in at most one $B_i \in \mathcal{B}$. Then, we can formulate the search problem as follows.

PROBLEM: CONSTRUCT STEINER TRIPLE SYSTEM
INSTANCE: v SUCH THAT $v \equiv 1, 3 \pmod 6$
FIND: MAXIMIZE $|\mathcal{B}|$
SUBJECT TO: $([1, v], \mathcal{B})$ IS A PARTIAL STEINER TRIPLE SYSTEM

The **universe** $\mathcal{X}$ is the set of all sets of blocks $\mathcal{B}$, such that $([1, v], \mathcal{B})$ is a partial Steiner triple system.
An **optimal solution** is any feasible solution with $|\mathcal{B}| = \frac{v(v-1)}{6}$.

DEFINITION. A point x is said to be a *live point* in $([1, v], \mathcal{B})$ if $r_x < \frac{v-1}{2}$. A pair $\{x, y\}$ is said to be a *live pair* in $([1, v], \mathcal{B})$ if there exists no $B \in \mathcal{B}$ with $\{x, y\} \subseteq B$

Stinson's hill-climbing algorithm for STSs

```
Algorithm Stinson's Algorithm(v)
    Numblocks ← 0
    V ← {1, 2, ... v}
    B ← ∅
    While (Numblocks < v(v−1)/2) do { Switch }
    output (V, B)


Algorithm Switch
    Chosse a random live point x.
    Choose random y, z such that
      {x, y} and {x, z} are live pairs.
    If ({y, z} is a live pair) then
       B ← B ∪ {{x, y, z}}
       Numblocks ← Numblocks +1
    else
        Let {w, y, z} ∈ B be the block containing {y, z}
        B ← B ∪ {{x, y, z}} \ {{w, y, z}}
```

See implementation details in the textbook.
Using appropriatte data structures, **Switch** is implemented in constant time.

Two heuristics for the Knapsack Problem

Knapsack (Optimization) Problem

Instance: Profits $p_0, p_1, \ldots, p_{n-1}$
Weights $w_0, w_1, \ldots, w_{n-1}$
Knapsack capacity M

Universe: $\mathcal{X} = \{0, 1\}^n$ (set of all n-tuples)
an n-tuple $[x_0, x_1, \ldots, x_{n-1}]$ is feasible if
$\Sigma_{i=0}^{n-1} w_i x_i \leq M$.

Objective: maximize $P(X) = \Sigma_{i=0}^{n-1} p_i x_i$.

A Simulated Annealing Algorithm for Knapsack

Neighbourhood function:

$$N(X) = N_1(X) = \{Y \in \{0,1\}^n : dist(X,Y) = 1\}$$

Algorithm $\textsc{KnapsackSimulatedAnnealing}(c_{max}, T_0, \alpha)$

```
c ← 0; T ← T_0;
X ← [x_0, x_1, ..., x_{n-1}] = [0, 0, ..., 0];
CurW ← 0; X_best ← X;
while (c ≤ c_max) do
        j ← randomInt(0, n − 1);
        Y ← X;
        y_j ← 1 − x_j;    (swap j coordinate of X)
        if (y_j = 1) and (curW + w_j > M) then Y ← fail;
        if (Y ≠ fail) then
           if (y_j = 1) then
              X ← Y;
              curW ← curW + w_j;
              if P(X) > P(X_best) then X_best ← X;
           else
                r ← random(0, 1);
                if (r < e^{−p_j/T}) then
                   X ← Y;
                   curW ← curW − w_j;
        c ← c + 1;
        T ← αT;
return (X_best);
```

Experimental results from Table 5.3 (page 178):

Tabu Search for Knapsack

We will use the same neighbourhood $N_1(.)$.

Do exhaustive search on the neighbourhood in order to find the best way to update the current solution.

Instead of Profit imporvement only, we look for improvements based on the ratio p_i/w_i:

1. Chose i with maximum p_i/w_i among the indexes j where $x_j = 0$, j is not on TABULIST, and changing x_j to 1 does not exceed M.
2. If there is no j as above, then choose i with minimum p_i/w_i among the indexes j where $x_j = 1$ and j is not on TABULIST.

This can be expressed by saying that we want to maximize

$$(-1)^{x_j}\frac{p_j}{w_j}.$$

Algorithm $\textsc{KnapsackTabuSearch}(c_{max}, L)$

$c \leftarrow 1$;
Select a random feasible $X = [x_0, x_1, \ldots, x_{n-1}] \in \{0,1\}^n$;
$curW \leftarrow \Sigma_{i=0}^{n-1} x_i w_i$;
$X_{best} \leftarrow X$;
while ($c \leq c_{max}$) do
    $N \leftarrow \{0, 1, \ldots, n-1\}$;
    $start \leftarrow max\{0, c - L\}$;
    for $j \leftarrow start$ to $c - 1$ do
        $N \leftarrow N \setminus \{\textsc{Tabulist}[j]\}$;
    for each ($i \in N$) do
        if ($x_i = 0$) and ($curW + w_i > M$) then $N \leftarrow N \setminus \{i\}$;
    if ($N = \emptyset$) then exit;
    Find $i \in N$ such that $(-1)^{x_i} p_i / w_i$ is maximum;
    $\textsc{Tabulist}[c] \leftarrow i$;
    $x_i \leftarrow 1 - x_i$;   (swap i coordinate)
    if ($x_i = 1$) then $curW \leftarrow curW + w_i$;
        else $curW \leftarrow curW - w_i$;
    if $P(X) > P(X_{best})$ then $X_{best} \leftarrow X$;
    $c \leftarrow c + 1$;
return X_{best};

Experimental results from Tables 5.4 and 5.6 (pages 180-181):

A Genetic Algorithm for the TSP

Traveling Salesman Problem (TSP)

Instance: a complete graph K_n
a cost function $c : V \times V \rightarrow R$
Find: a Hamiltonian circuit $[x_0, x_1, \ldots, x_{n-1}]$ that minimizes
$C(X) = c(x_0, x_1) + c(x_1, x_2) + \ldots + c(x_{n-1}, x_0)$

Note that $2n$ permutations represent the same cycle.

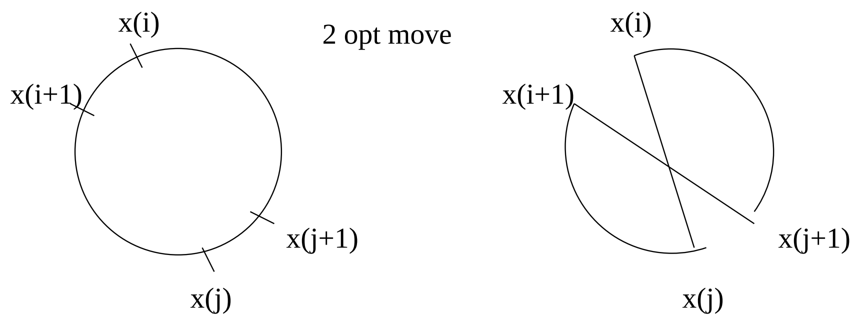
Universe: $\mathcal{X}$ = set of all $n!$ permutations.

Steps:

- Selection of initial population.
- Mutation: steepest ascent 2-opt.
- Recombination using two methods: partially matched crossover and another method.

Mutation

Steepest ascent algorithm based on the 2-opt heuristic:



Gain in applying a 2-opt move:

$$\begin{aligned} G(X,i,j) &= C(X) - C(X_{ij}) \\ &= c(x_i, x_{i+1}) + c(x_j, x_{j+1}) - c(x_{i+1}, x_{j+1}) - c(x_i, x_j) \end{aligned}$$

$N(X)$ = all $Y \in \mathcal{X}$ that can be obtained from X by a 2-opt move.

Algorithm SteepestAscentTwoOpt(X)

$done \leftarrow false$;
while ($not\ done$) do
  $done \leftarrow true$; $g_0 \leftarrow 0$;
  for $i \leftarrow 0$ to $n-1$ do
    for $j \leftarrow i+2$ to $n-1$ do
      $g \leftarrow G(X,i,j)$;
      if ($g > g_0$) then
        $g_0 \leftarrow G$; $i_0 \leftarrow i$; $j_0 \leftarrow j$;
  if ($g_0 > 0$) then
    $X \leftarrow X_{i_0,j_0}$;
    $done \leftarrow false$;

Selecting the initial population

Randomly pick one and then mutate it:

Algorithm SELECT($popsize$)
for $i \leftarrow 0$ to $popsize - 1$ do
$r \leftarrow$ RANDOMINTEGER($0, n! - 1$);
$P_i \leftarrow$ PERMLEXUNRANK(n, r);
STEEPESTASCENTTWOOPT(P_i);
return $[P_0, P_1, \ldots, P_{popsize-1}]$;

Two recombination algorithms

1. Partially Matched Crossover

Algorithm PMREC(A, B)
$h \leftarrow$ RANDOMINTEGER($10, n/2$); (length of the substring)
$j \leftarrow$ RANDOMINTEGER($0, n - 1$); (start of the substring)
$(C, D) \leftarrow$ PARTIALLYMATCHEDCROSSOVER($A, B, j, (h + j) mod\ n$)
STEEPESTASCENTTWOOPT(C);
STEEPESTASCENTTWOOPT(D);
return (C, D);

2. Another Recombination Algorithm

Algorithm MGKREC(A, B)

$h \leftarrow$ RANDOMINTEGER$(10, n/2)$; (length of the substring)
$j \leftarrow$ RANDOMINTEGER$(0, n-1)$; (start of the substring)
$T \leftarrow \emptyset$;
(pick subcycle of length h starting from pos j:)
for $i \leftarrow 0$ to $h-1$ do
    $D[i] \leftarrow B[(i+j) mod n]$;
    $T \leftarrow T \cup \{D[i]\}$;
Complete cycle with permutation in A using guys not already in D in the order prescribed by A:
for $j \leftarrow 0$ to $n-1$ do
    if $A[j] \notin T$ then $D[i] \leftarrow A[j]$;
        $i \leftarrow i+1$;
STEEPESTASCENTTWOOPT(D);
(Similarly build C swapping A and B roles:)
$j \leftarrow$ RANDOMINTEGER$(0, n-1)$; (start of the substring)
$T \leftarrow \emptyset$;
for $i \leftarrow 0$ to $h-1$ do
    $C[i] \leftarrow A[(i+j) mod n]$;
    $T \leftarrow T \cup \{C[i]\}$;
for $j \leftarrow 0$ to $n-1$ do
    if $B[j] \notin T$ then $C[i] \leftarrow B[j]$;
        $i \leftarrow i+1$;
STEEPESTASCENTTWOOPT(C);
return (C, D);

Genetic Algorithm for TSP

Algorithm GeneticTSP($popsize, c_{max}$)

$c \leftarrow 1$;

$[P_0, P_1, \ldots, P_{popsize-1}] \leftarrow$Select($popsize$);

Sort $P_0, P_1, \ldots, P_{popsize-1}$ in increasing order of cost.

$X_{best} \leftarrow P_0$;

$BestCost \leftarrow C(P_0)$;

while ($c \leq c_{max}$) do

for $i \leftarrow 0$ to $popsize/2 - 1$ do

$(P_{popsize+2i}, P_{popsize+2i+1}) \leftarrow$ Rec (P_{2i}, P_{2i+1});

Sort $P_0, P_1, \ldots, P_{2popsize-1}$ in increasing order of cost.

$curCost \leftarrow C(P_0)$;

if ($curCost < BestCost$) then

$X_{best} \leftarrow P_0$;

$BestCost \leftarrow curCost$;

$c \leftarrow c + 1$;

return X_{best};

Note: Rec represents either of the two recombination algorithms.

Experimental results from Tables 5.7 and 5.8 (pages 186-187):