CSI5165 COMBINATORIAL ALGORITHMS

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INTRODUCTION TO COMBINATORIAL ALGORITHMS

Introduction to Combinatorial Algorithms

What are :

- Combinatorial Structures?
- Combinatorial Algorithms?
- Combinatorial Problems?

Combinatorial Structures

Combinatorial structures are *collections* of k-subsets/K-tuple/permutations from a parent set (finite).

Examples:

- **Undirected Graphs:**
  Collections of 2-subsets (edges) of a parent set (vertices).

$$V = \{1,2,3,4\} \quad E = \{\{1,2\},\{1,3\},\{1,4\},\{3,4\}\}$$

- **Directed Graphs:**
  Collections of 2-tuples (directed edges) of a parent set (vertices).

$$V = \{1,2,3,4\} \quad E = \{(2,1),(3,1),(1,4),(3,4)\}$$

- **Hypergraphs or Set Systems:**
  Similar to graphs, but (hyper) edges may be sets with more than two elements.

$$V = \{1,2,3,4\} \quad E = \{\{1,3\},\{1,2,4\},\{3,4\}\}$$

Building blocks: finite sets, finite lists (tuples)

- Finite Sets

  $X = \{1, 2, 3, 5\}$

  – undordered structure, no repeats
  $\{1, 2, 3, 5\} = \{2, 1, 5, 3\} = \{2, 1, 1, 5, 3\}$
  – cardinality (size) = number of elements $|X| = 4$.

  A <u>k-subset</u> of a finite set X is a set $S \subseteq X$, $|S| = k$.
  For example: $\{1, 2\}$ is a 2-subset of X.

- Finite Lists (or Tuples)

  $L = [1, 5, 2, 1, 3]$

  – ordered structure, repeats allowed
  $[1, 5, 2, 1, 3] \neq [1, 1, 2, 3, 5] \neq [1, 2, 3, 5]$
  – length = number of items, length of L is 5.

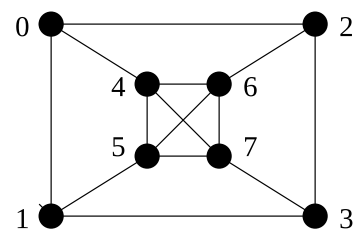
  An <u>n-tuple</u> is a list of length n.
  A <u>permutation</u> of an n-set X is a list of length n such that every element of X occurs exactly once.

$$X = \{1, 2, 3\}, \quad \pi_1 = [2, 1, 3] \quad \pi_2 = [3, 1, 2]$$

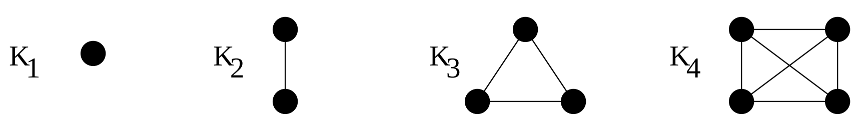
Graphs

Definition. A *graph* is a pair (V, E) where:
V is a finite set (of vertices).
E is a finite set of 2-subsets (called edges) of V.

Example: G_1 : $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$
$E = \{\{0, 4\}, \{0, 1\}, \{0, 2\}, \{2, 3\}, \{2, 6\}, \{1, 3\}, \{1, 5\}, \{3, 7\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}\}$



Complete graphs: graphs with all possile edges.
Examples:



Substructures of a graph:

1. A hamiltonian circuit (hamiltonian cycle) is a closed path that passes through each vertex once.

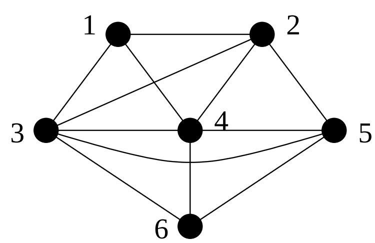
   The following list describes a hamiltonian cycle in G_1:
   $[0, 1, 5, 4, 6, 7, 3, 2]$ (different lists may describe the same cycle).

Traveling Salesman Problem: given a weight/cost function $w : E \to R$ on the edges of G, find a smallest weight hamiltonian cycle in G.

2. A clique in a graph $G = V, E)$ is a subset $C \subseteq V$ such that $\{x, y\} \in E$, for all $x, y \in C$ with $x \neq y$.
(Or equivalently: the subgraph induced by C is complete).

Example:

G_2 :



Some cliques of G_2:

Maximum cliques of G_2:

Famous problems involving cliques:

- Maximum clique problem: find a maximum clique in a graph.
- All cliques problem: find all cliques in a graph without repetition.

Set systems/Hypergraphs

DEFINITION. A *set system (or hypergraph)* is a pair $(X, \mathcal{B})$ where:
X is a finite set (of points).
$\mathcal{B}$ is a finite set of subsets of X (blocks).

Examples:

- Graph: A graph is a set system with every block with cardinality 2.
- Partition of a finite set:

  A partition is a set system $(X, \mathcal{B})$ such that
  $B_1 \cap B_2 = \emptyset$ for all $B_1, B_2 \in \mathcal{B}, B_1 \neq B_2$, and

  $$\cup_{B \in \mathcal{B}} B = X.$$

- Steiner triple system (a type of combinatorial designs):

  $\mathcal{B}$ is a set of 3-subsets of X such that for each $x, y \in X, x \neq y$, there exists eactly one block $B \in \mathcal{B}$ with $\{x, y\} \subseteq B$.

  Example:
  $X = \{0, 1, 2, 3, 4, 5, 6\}$
  $\mathcal{B} =$
  $\{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$

Combinatorial Algorithms

Algorithms for investigating combinatorial structures. Three types:

- **Generation**
  Construct all combinatorial structures of a particular type.
  - Generate all subsets/permutations/partitions of a set.
  - Generate all cliques of a graph.
  - Generate all maximum cliques of a graph.
  - Generate all Steiner triple systems of a finite set.
- **Enumeration**
  Compute the number of different structures of a particular type.
  - Compute the number of subsets/permutat./partitions of a set.
  - Compute the number of cliques of a graph.
  - Compute the number of maximum cliques of a graph.
  - Compute the number of Steiner triple systems of a finite set.
- **Search**
  Find at least one example of a combinatorial structures of a particular type (if one exists).
  Optimization problems can be seen as a type of search problem.
  - Find a Steiner triple system on a finite set. (feasibility)
  - Find a maximum clique of a graph. (optimization)
  - Find a hamiltonian cycle in a graph. (feasibility)
  - Find a smallest weight hamiltonian cycle in a graph. (optimization)

Hardness of Search and Optimization

Many search and optimization problems are NP-hard, which means that

unless $P = NP$ (an important unsolved complexity question)

no polytomial-time algorithm to solve the problem would exist.
Approaches for dealing with NP-hard problems:

- Exhaustive Search
  - exponential-time algorithms.
  - solves the problem exactly

  (Backtracking and Branch-and-Bound)
- Heuristic Search
  - algorithms that explore a search space to find a feasible solution that is "close to" optimal, within a time limit
  - approximates a solution to the problem

  (Hill-climbing, Simulated annealing, Tabu-Search, Genetic Algorithms)
- Approximation Algorithms
  - polynomial time algorithm
  - we have a provable guarantee that the solution found is "close to" optimal.

  (not covered in this course)

Types of Search Problems

1) **Decision Problem**:
A yes/no problem

Problem 1:
Clique (decision)
Instance: graph $G = (V, E)$,
target size k

Question:
Does there exist a clique C of G with $|C| = k$?

2) **Search Problem**:
Find the guy.

Problem 2:
Clique (search)
Instance: graph $G = (V, E)$,
target size k

Find:
a clique C of G with $|C| = k$, if one exists.

3) **Optimal Value**:
Find the largest target size.

Problem 3:
Clique (optimal value)
Instance: graph $G = (V, E)$,

Find:
the maximum value of $|C|$, where C is a clique

4) **Optimization**:
Find an optimal guy.

Problem 4:
Clique (optimization)
Instance: graph $G = (V, E)$,

Find:
a clique C such that $|C|$ is maximize (max. clique)

Plan for the Course

1. **Generating elementary combinatorial objects**

   Sequential generation (successor), rank, unrank.
   Algorithms for subsets, k-subsets, permutations.
   Reference: textbook chapter 2. [2 weeks]

2. **Exhaustive Generation and Exhaustive Search**

   Backtracking algorithms
   (exhaustive generation, exhaustive search, optimization)

   Brach-and-bound
   (exhaustive search, optimization)
   Reference: textbook chapter 4. [3 weeks]

3. **Heuristic Search**

   Hill-climbing, Simulated annealing, Tabu-Search, Genetic Algs.
   Applications of these techniques to various problems.

   Reference: textbook chapter 5. [3 weeks]

4. **Computing Isomorphism and Isomorph-free Exhaustive Generation**

   Graph isomorphism, isomorphism of other structures.
   Computing invariants.
   Computing certificates.
   Isomorph-free exhaustive generation.
   Example: Generate all trees on n vertices, without isomorphic copies.

   Reference: textbook chapter 7, papers. [3 weeks]