SOME

EXAMPLES

→

# EXAMPLE APPLICATION OF THEOREM 4

$$\boxed{\begin{array}{c} a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \\ a_0 = 2,\ a_1 = 5,\ a_2 = 15 \end{array}}$$

characteristic polynomial is

$$r^3 - 6r^2 + 11r + 6$$

the characteristic roots are $r_1 = 1$, $r_2 = 2$, $r_3 = 3$

since $r^3 - 6r^2 + 11r + 6 = (r-1)(r-2)(r-3)$

So the solution has the form

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$$

find $\alpha_0, \alpha_2, \alpha_3$ using initial conditions:

$$\begin{cases} a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3 \\ a_1 = 5 = 1\alpha_1 + 2\alpha_2 + 3\alpha_3 \\ a_2 = 15 = 1\alpha_1 + 4\alpha_2 + 9\alpha_3 \end{cases}$$

Solve these 3 simultaneous equations:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 5 \\ 1 & 4 & 9 & 15 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 6 & 10 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

$$\left.\begin{array}{r} \alpha_1 + \alpha_2 + \alpha_3 = 2 \\ \alpha_2 + 3\alpha_3 = 5 \\ 2\alpha_3 = 4 \end{array}\right\} \quad \begin{array}{l} \alpha_1 = 1 \\ \alpha_2 = -1 \\ \alpha_3 = 2 \end{array}$$

$$\boxed{a_n = 1.1^n - 1.2^n + 2.3^n}$$

# EXAMPLE APPLICATION OF THEOREM 4

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \quad \text{with } a_0 = 1,\ a_1 = -2,\ a_2 = -1$$

degree $k=3$

characteristic polynomial

$x^3 + 3x^2 + 3x + 1$ (since $c_1 = -3$, $c_2 = -3$, $c_3 = -1$)

$= (x+1)^3 \Rightarrow r_0 = -1$ with multiplicity $m = 3$

Solution is of the form:

$$a_n = (\alpha_0 + \alpha_1 n + \alpha_2 n^2)(-1)^n$$

Use initial conditions to determine $\alpha_0, \alpha_1, \alpha_2$

$$a_0 = 1 = (\alpha_0 + \alpha_1 \cdot 0 + \alpha_2 0^2)(-1)^0$$
$$a_1 = -2 = (\alpha_0 + \alpha_1 1 + \alpha_2 1^2)(-1)^1$$
$$a_2 = -1 = \underbrace{(\alpha_0 + \alpha_1 2 + \alpha_2 2^2)}_{k=3}\underbrace{(-1)^2}_{r_0=-1}$$

$$\begin{cases} \boxed{\alpha_0 = 1} \\ -\alpha_0 - \alpha_1 - \alpha_2 = -2 \\ \alpha_0 + 2\alpha_1 + 4\alpha_2 = -1 \end{cases} \quad \begin{array}{l} -\alpha_1 - \alpha_2 = -1 \\ 2\alpha_1 + 4\alpha_2 = -2 \end{array}$$

$\alpha_2 = 1 - \alpha_1$ $\quad \boxed{\alpha_2 = -2}$

$2\alpha_1 + 4(1-\alpha_1) = -2$

$-2\alpha_1 = -6$

$\boxed{\alpha_1 = 3}$

SOLUTION TO EQUATION: $\alpha_0 = 1$, $\alpha_1 = 3$, $\alpha_2 = -2$

SOLUTION OF RECURRENCE RELATION:

$$a_n = (1 + 3n - 2n^2) \times (-1)^n$$

# Solving Hanoi Tower Rec. Relation

$$\begin{cases} H_1 = 1 \\ H_n = 2H_{n-1} + 1 \end{cases}$$

Using theorem 6 to find a particular solution:

$k=1$ $\boxed{t=0,\ s=1,\ m=1}$

characteristic polynomial: $p(x) = x-2$

roots of characteristic polynomial:

$r-2=0 \Rightarrow \boxed{r=2}$

Sol. to homogeneous equation:

$\boxed{a_n^h = \alpha_1 2^n}$

$\underline{s \neq r}$

particular solution:

$a_n^p = p_0 \cdot s^n = p_0 1^n = p_0$

Using $H_n = 2H_{n-1} + 1$ (once since $t=1$)

$p_0 = 2p_0 + 1 \Rightarrow \boxed{p_0 = -1}$

particular solution

$\boxed{a_n^p = -1}$

Sol of the form

$H_n = a_n^p + a_n^h = -1 + \alpha_1 2^n$

---

Using inicial condition:

$H_1 = 1 = -1 + 2\alpha_1 \Rightarrow \boxed{\alpha_1 = 1}$

$H_n = -1 + 1.2^n$

$\boxed{H_n = 2^n - 1}$

IMPORTANT! INITIAL SOLUTION USED AT THE VERY END.

homo: $a_n = 6a_{n-1} - 9a_{n-2}$ $\quad r^2 - 6r + 9 = 0$ $\boxed{r_1 = 3}$ multipl $\boxed{m=2}$

$F(n) = (b_t n^t + \dots + b_1 n + b_0) s^n$

$a_n = 6a_{n-1} - 9a_{n-2} + 3^n$

s is root with $m=2$

$t=0$

$$\boxed{a_n^p = n^2 (p_0) 3^n}$$

finding $p_0$, apply rec. once:

$a_2 = 6a_1 - 9a_0 + 3^2$

$2^2 p_0 . 3^2 = 6(1^2 p_0 . 3^1) - 9(0^2 p_0 3^0) + 9$

$36 p_0 = 18 p_0 + 9 \Rightarrow 2p_0 = 9$ $\boxed{p = 9/2}$

$$\boxed{a_n^p = n^2 \frac{9}{2} 3^n}$$

$a_n = 6a_{n-1} - 9a_{n-2} + n3^n$

s is root with $m=2$

$t=1$

$$\boxed{a_n^p = n^2 (p_1 n + p_0) 3^n}$$

$a_n = 6a_{n-1} - 9a_{n-2} + n^2 2^n$

s is not root

$t=2$

$$\boxed{a_n^p = (p_2 n^2 + p_1 n + p_0) 2^n}$$

$a_n = 6a_{n-1} - 9a_{n-2} + (n^2+1)3^n$

s is root with $m=2$

$t=2$

$$\boxed{a_n^p = n^2 (p_2 n^2 + p_1 n + p_0) 3^n}$$