What is Mathematics, really?

- It's *not* just about numbers!
- Mathematics is *much* more than that:

> Mathematics is, most generally, the study of any and all *absolutely certain* truths about any and all *perfectly well-defined* concepts.

- But, these concepts can be *about* numbers, symbols, objects, images, sounds, *anything*!

What are "discrete structures" anyway?

- "***Discrete***" (≠ "discreet"!) - Composed of distinct, separable parts. (Opposite of *continuous*.)

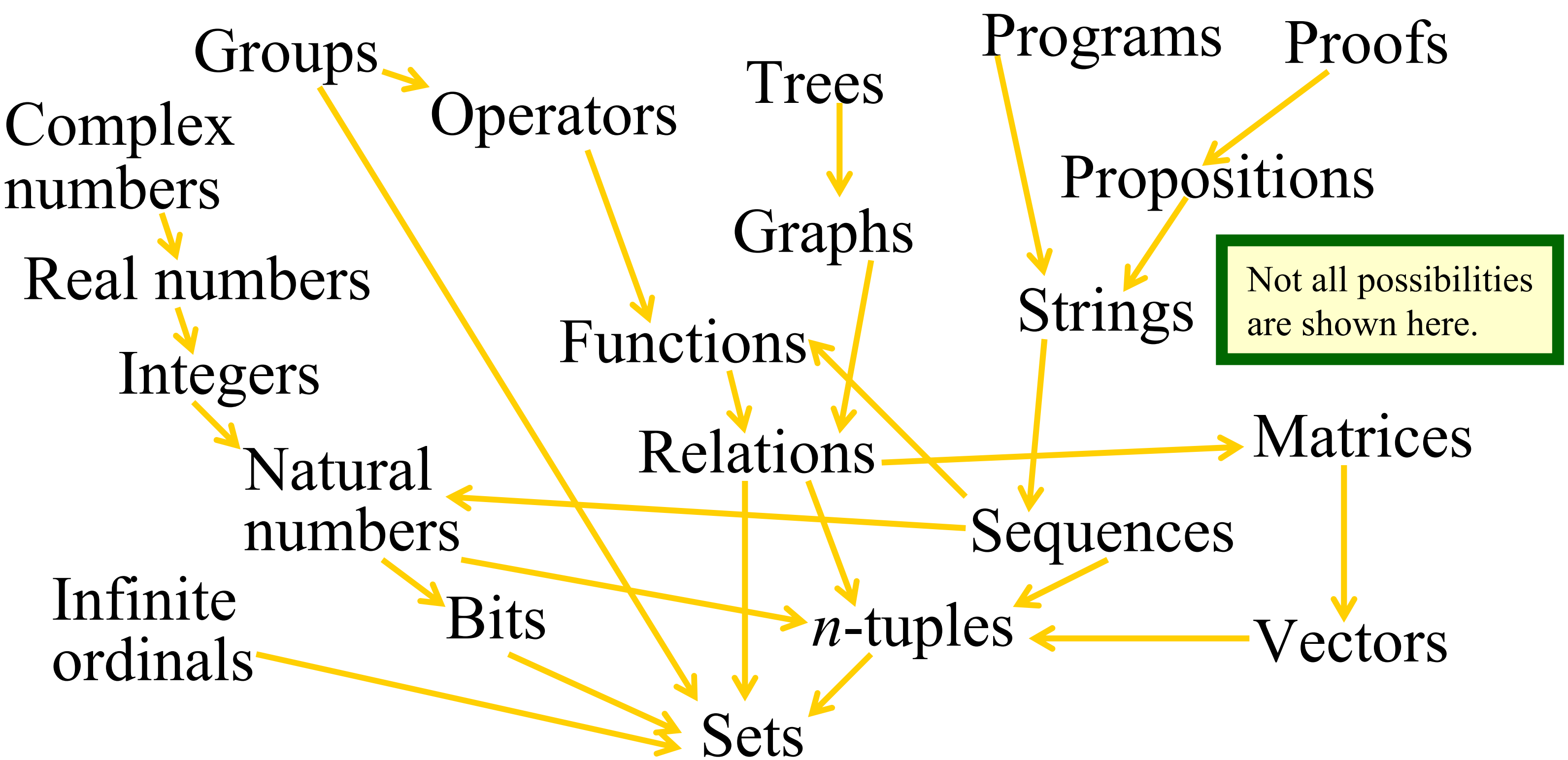
  "***Structures***" - Objects built up from simpler objects according to some definite pattern.
- "***Discrete Mathematics***" - The study of discrete, mathematical objects and structures.

Discrete Structures you will Study in your program

- Propositions
- Predicates
- Proofs
- Sets
- Functions
- Orders of Growth
- Algorithms
- Integers
- Summations
- Sequences
- Strings
- Permutations
- Combinations
- Relations
- Graphs
- Trees
- Logic Circuits
- Automata

Relationships Between Structures

- "$\rightarrow$" :≝ "Can be defined in terms of"



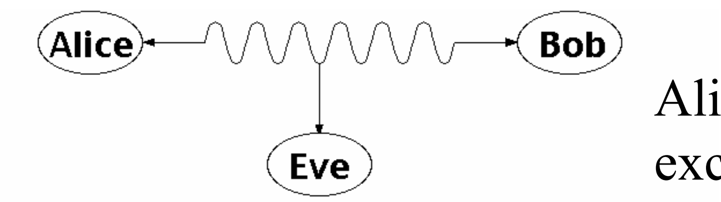
Why Study Discrete Math?

- The basis of all of digital information processing is: *Discrete manipulations of discrete structures represented in memory.*
- It's the basic language and conceptual foundation for all of computer science.
- Discrete math concepts are also widely used throughout math, science, engineering, economics, biology, *etc.*, …
- A generally useful tool for rational thought!

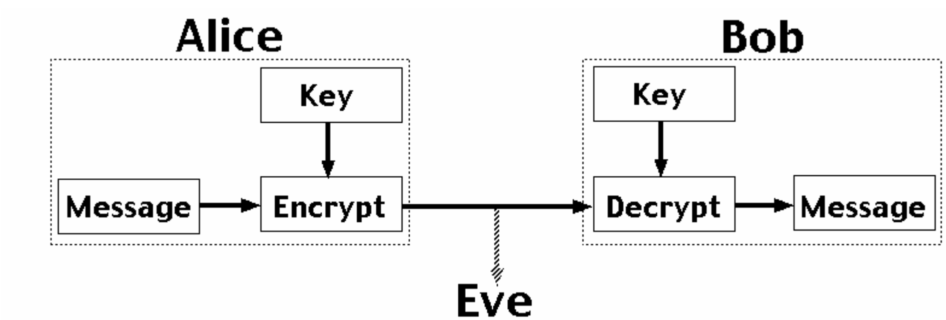
Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, *etc.*…
- *I.e.*, the whole field!

Number Theory: RSA and Public-key Cryptography

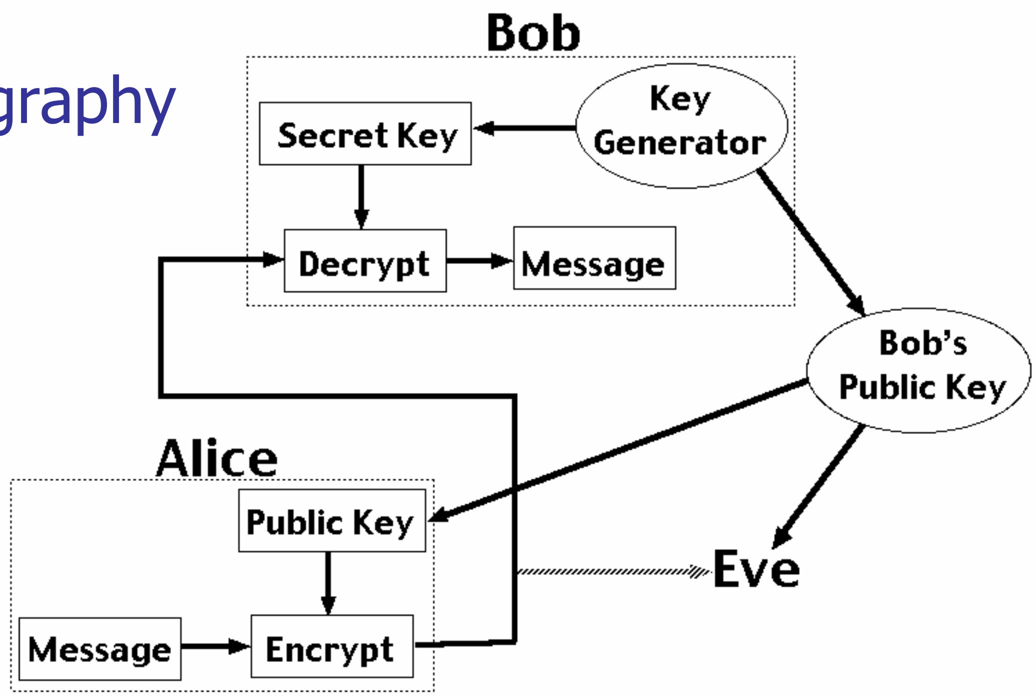


Alice and Bob have never met but they would like to exchange a message. Eve would like to eavesdrop.



They could come up with a good encryption algorithm and exchange the encryption key – but how to do it without Eve getting it? (If Eve gets it, all security is lost.)

Number Theory:
Public-key Cryptography



RSA – Public Key Cryptosystem (why RSA?)

Uses modular arithmetic and large primes → Its security comes from the computational difficulty of factoring large numbers.

RSA Approach

Encode:

$C = M^e \pmod{n}$

M is the plaintext; C is ciphertext

n = pq with p and q large primes (e.g. 200 digits long!)

e is relative prime to (p-1)(q-1)

Decode:

$C^d = M \pmod{pq}$

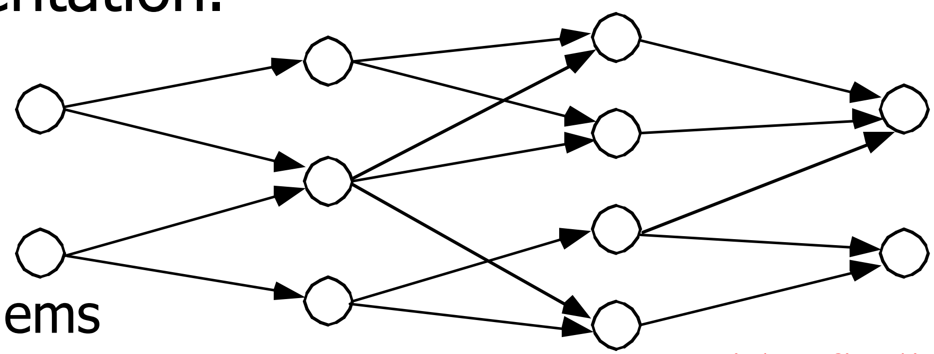
d is inverse of e modulo (p-1)(q-1)

What does this all mean??
How does this actually work?
Not to worry. We'll find out.

The process of encrypting and decrypting a message correctly results in the original message (and it's fast!)

Graphs and Networks

Many problems can be represented by a graphical network representation.



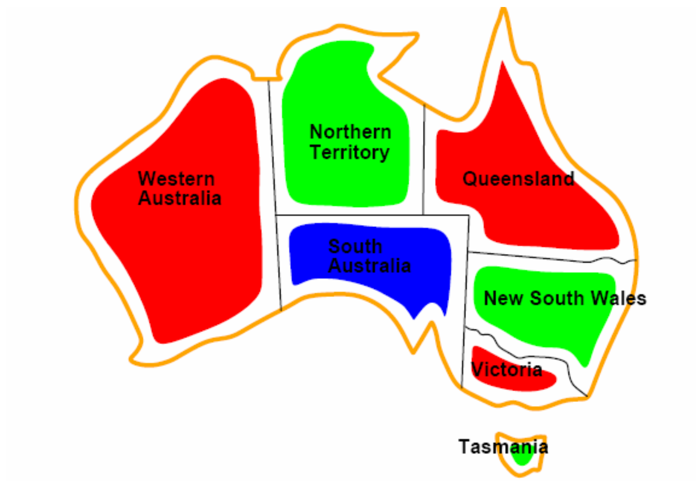
- Examples:
  - Distribution problems
  - Routing problems
  - Maximum flow problems
  - Designing computer/phone / road networks
  - Equipment replacement
  - And of course the Internet

Aside: finding the right problem representation is one of the key issues in this course.

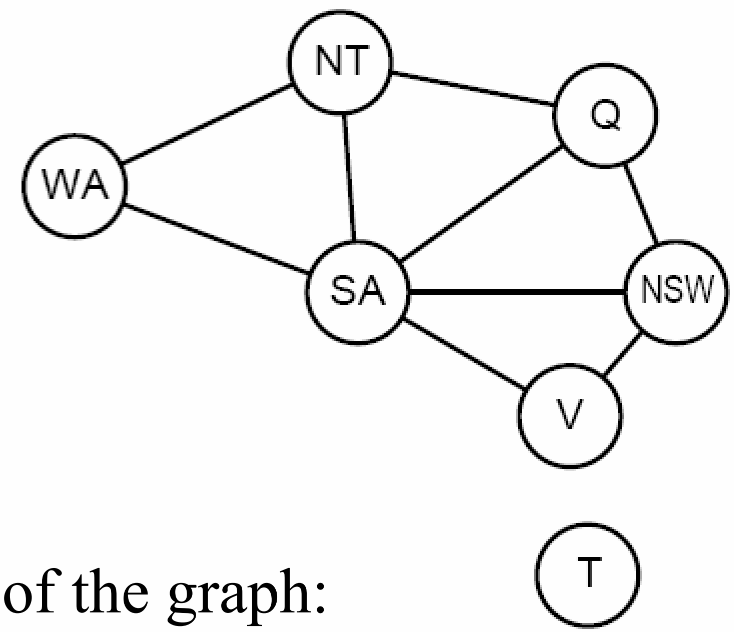
Applications of Networks

Applications	Physical analog of nodes	Physical analog of arcs	Flow
Communication systems	phone exchanges, computers, transmission facilities, satellites	Cables, fiber optic links, microwave relay links	Voice messages, Data, Video transmissions
Hydraulic systems	Pumping stations Reservoirs, Lakes	Pipelines	Water, Gas, Oil, Hydraulic fluids
Integrated computer circuits	Gates, registers, processors	Wires	Electrical current
Mechanical systems	Joints	Rods, Beams, Springs	Heat, Energy
Transportation systems	Intersections, Airports, Rail yards	Highways, Airline routes Railbeds	Passengers, freight, vehicles, operators

Example: Coloring a Map: How to color this map so that no two adjacent regions have the same color?



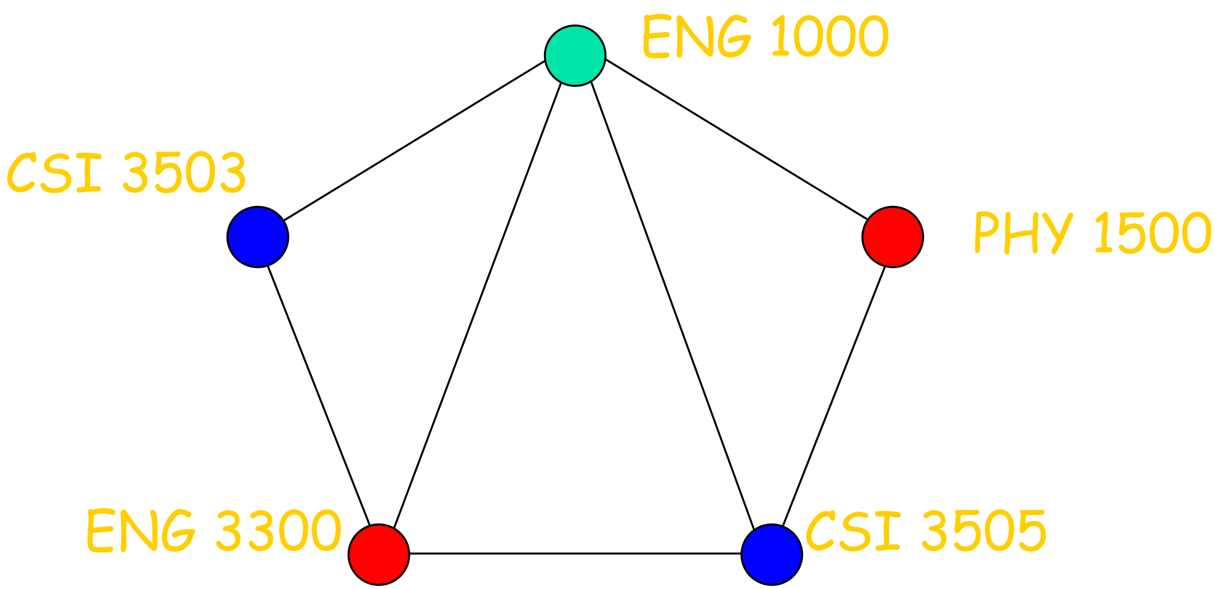
Graph representation



Coloring the nodes of the graph:
What's the minimum number of colors such that any two nodes connected by an edge have different colors?

- The Four Color Theorem – ***Any planar graph can be properly colored by at most four colors.***

- Proof: Appel and Haken 1976; careful case analysis performed by computer; proof reduced the infinitude of possible maps to 1,936 reducible configurations (later reduced to 1,476) which had to be checked one by one by computer. The computer program ran for hundreds of hours.
- The first significant *computer-assisted* mathematical proof.
- *Write-up was hundreds of pages including code!*


ENG 1000
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PHY 1500
ENG 3300
CSI 3505

Scheduling of Final Exams: How can the final exams at UofO be scheduled so that no student has two exams at the same time? (Note not obvious this has anything to do with graphs or graph coloring.)

Graph:

A vertex correspond to a course.

An edge between two vertices denotes that there is at least one common student in the courses they represent.

Each time slot for a final exam is represented by a different color.

A coloring of the graph corresponds to a valid schedule of the exams.

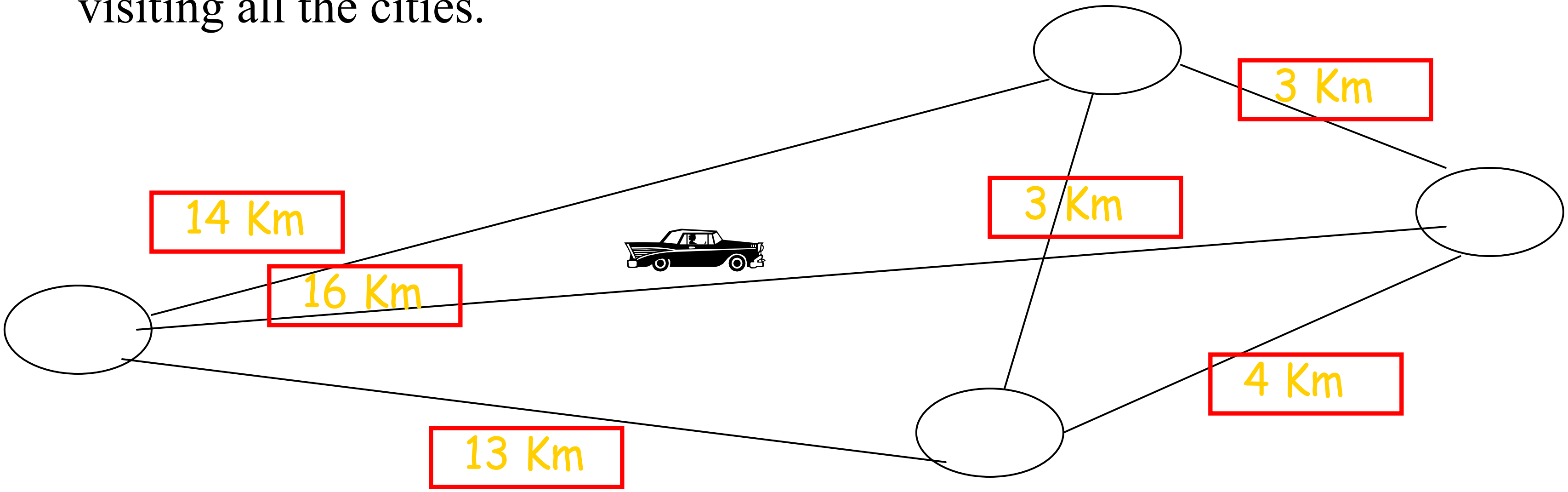
Frequency Assignments: T.V. channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled as a graph coloring?

- A vertex corresponds to one station;
- There is a edge between two vertices if they are located within 150 miles of each other
- Coloring of graph --- corresponds to a valid assignment of channels; each color represents a different channel.

Index Registers: In efficient compilers the execution of loops can be speeded up by storing frequently used variables temporarily in registers in the central processing unit, instead of the regular memory. For a given loop, how many index registers are needed?

- Each vertex corresponds to a variable in the loop.
- An edge between two vertices denotes the fact that the corresponding variables must be stored in registers at the same time during the execution of the loop.
- Chromatic number of the graph gives the number of index registers needed.

Traveling Salesman: Find a closed tour of minimum length visiting all the cities.

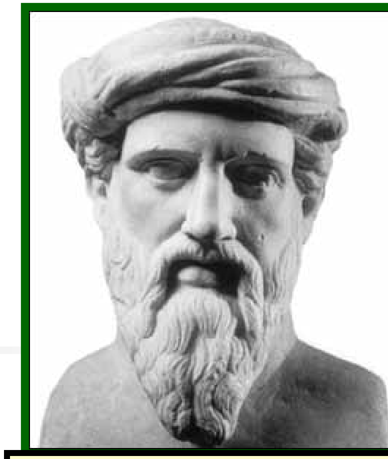


TSP → lots of applications:

Transportation related: scheduling deliveries

Many others: e.g., Scheduling of a machine to drill holes in a circuit board ; Genome sequencing; etc

- Upon completion of this course, the student should be able to:
  - Check validity of simple logical arguments (proofs).
  - Check the correctness of simple algorithms.
  - Creatively construct simple instances of valid logical arguments and correct algorithms.
  - Describe the definitions and properties of a variety of specific types of discrete structures.
  - Correctly read, represent and analyze various types of discrete structures using standard notations.



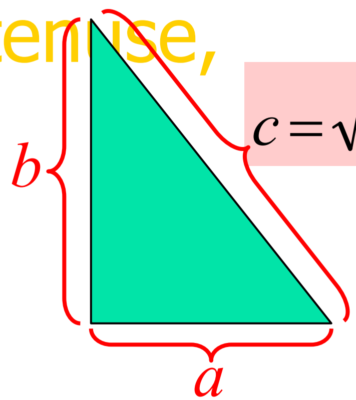
Pythagoras of Samos
(ca. 569-475 B.C.)

A Proof Example

- **Theorem:**

(Pythagorean Theorem of Euclidean geometry)

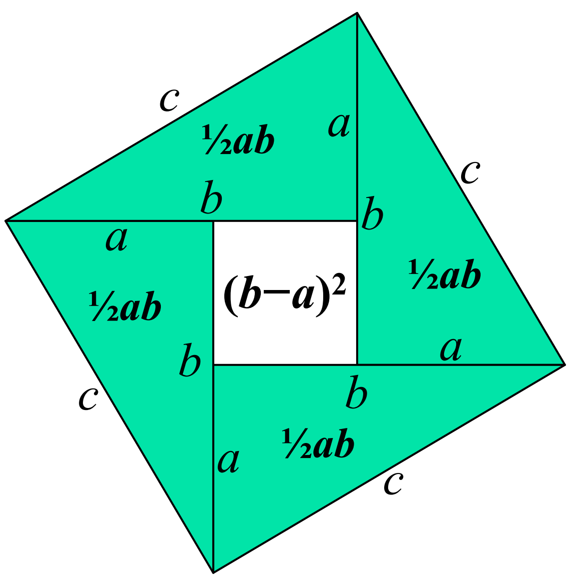
For <u>any</u> real numbers *a*, *b*, and *c*, if *a* and *b* are the base-length and height of a right triangle, and *c* is the length of its hypotenuse, then $a^2 + b^2 = c^2$.



$$c = \sqrt{a^2 + b^2}$$

- **Proof:** See next slide.

- **Proof.** Consider the below diagram:
- Exterior square area = c^2, the sum of the following regions:
  - The area of the 4 triangles = $4(\frac{1}{2}ab) = 2ab$
  - The area of the small interior square = $(b-a)^2 = b^2-2ab+a^2$.
- Thus, $c^2 = 2ab + (b^2-2ab+a^2) = a^2 + b^2$. ■



Note: It is easy to show that the exterior and interior quadrilaterals in this construction are indeed squares, and that the side length of the internal square is indeed $b-a$ (where b is defined as the length of the longer of the two perpendicular sides of the triangle). These steps would also need to be included in a more complete proof.

Areas in this diagram are in boldface; lengths are in a normal font weight.