CSI 2101 Discrete Structures — Winter 2012
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Quiz #8

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1. Find the form of the particular solution for the following functions $F(n)$ that form part of a nonhomogeneous recurrence relation, assuming that there is no overlap with roots of the associated homogeneous relation.

   Remember **Section 7.2 Theorem 6** from the text, which says that if the function $F(n)$ has the form:

   $$F(n) = (\underbrace{b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0}_{\text{polynomial part}}) \underbrace{s^n}_{\text{exponential part}}$$

   and s is not a root of the associated homogeneous relation, then the particular solution has form:

   $$a_n^{(p)} = (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

   where $p_0, \ldots, p_t$ are real numbers.

   (a) $F(n) = n^2$

   Here, the polynomial part is n^2, so $t = 1$, and there is no exponential part, which is the same as having an exponential part $1 = 1^n$, so $s = 1$. Thus, using the formula, the particular solution has the following form:

   $$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 1^n = p_2 n^2 + p_1 n + p_0.$$

   (b) $F(n) = n3^n$

   Here, the polynomial part is n, so $t = 1$, and the exponential part is 3^n, so $s = 3$. Thus, using the formula, our particular solution is:

   $$a_n^{(p)} = (p_1 n + p_0) 3^n.$$

2. **Homework S7.2#28:** Find all solutions to the recurrence relation $a_n = 2a_{n-1} + 2n^2$ with initial value $a_1 = 4$.

   This is a nonhomogeneous recurrence relation, so we need to find the solution to the associated homogeneous relation and a particular solution.

The associated homogeneous relation is:

$$a_n = 2a_{n-1}.$$

This has characteristic equation $r - 2 = 0$, which only has one root, namely $r = 2$. Thus, the solution to the homogeneous relation has the form:

$$a_n^{(h)} = \alpha 2^n$$

for some constant α (which we will find later using the initial value).

We now find the particular solution. We have that $F(n) = 2n^2$, so this function, by Theorem 6, is just a polynomial in n of degree 2, giving that $t = 2$ and $s = 1$, so since $s \neq 2$, we get that the particular solution has the form:

$$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0)1^n = p_2 n^2 + p_1 n + p_0.$$

To find the values of p_0, p_1, p_2, we substitute the particular solution into the original recurrence relation:

$$\begin{aligned} a_n &= 2a_{n-1} + 2n^2 \\ (p_2 n^2 + p_1 n + p_0) &= 2(p_2(n-1)^2 + p_1(n-1) + p_0) + 2n^2 \\ (p_2 + 2)n^2 + (p_1 - 4p_2)n + (2p_2 - 2p_1 + p_0) &= 0 \\ (p_2 + 2)n^2 + (p_1 - 4p_2)n + (2p_2 - 2p_1 + p_0) &= 0n^2 + 0n + 0 \end{aligned}$$

This gives us three equations by setting the coefficients of n^2 to be equal:

$$p_2 + 2 = 0$$

and the coefficients of n to be equal:

$$p_1 - 4p_2 = 0$$

and the coefficients of the constant part to be equal:

$$2p_2 - 2p_1 + p_0 = 0$$

We solve this system to get that $p_0 = -12$, $p_1 = -8$, and $p_2 = -2$, so the particular solution is:

$$a_n^{(p)} = -2n^2 - 8n - 12.$$

We then have that the general solution is the sum of the homogeneous solution and the particular solution:

$$\begin{aligned} a_n &= a_n^{(h)} + a_n^{(p)} \\ &= \alpha 2^n - 2n^2 - 8n - 12 \end{aligned}$$

We can now solve for α using the initial value $a_1 = 4$:

$$a_1 = 4 = \alpha 2^1 - 2(1)^2 - 8(1) - 12$$

which gives that $\alpha = 13$. The final solution is then:

$$a_n = 13 \cdot 2^n - 2n^2 - 8n - 12.$$

Note that you can check your solution is correct by calculating the first few values a_1, a_2, a_3, etc and making sure that your final solution gives the same values as the original recurrence relation.