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ON THE BOOK EMBEDDING OF ORDERED SETS

ABSTRACT. In the book embedding of an ordered set, the elements of the set are embedded along the spine of a book to form a linear extension. The pagenumber (or stack number) is the minimum number of pages needed to draw the edges as simple curves such that edges drawn on the same page do not intersect. The pagenumber problem for ordered sets is known to be NP-complete, even if the order of the elements on the spine is fixed. In this paper, we investigate this problem for some classes of ordered sets. We provide an efficient algorithm for embedding bipartite interval orders in a book with the minimum number of pages. We also give an upper bound for the pagenumber of general bipartite ordered sets and the pagenumber of complete multipartite ordered sets. At the end of this paper we discuss the effect of a number of diagram operations on the pagenumber of ordered sets. We give an answer to an open question by Nowakowski and Parker [7] and we provide several known and new open questions we consider worth investigating.

1. INTRODUCTION

Data processing has become one of the cornerstones of information technology. With always more data being stored, simulated and analyzed, the attention paid to tools to manipulate this data has increased. Visualization of complex structures is one such tool, so a wide range of data visualization techniques have been developed. Each visualization technique is typically better suited for some particular structures (a tree, a planar graph, a lattice etc.). One of these complex structures is that of an *ordered set*, that is, a set of elements together with a reflexive, antisymmetric and transitive relation on these elements. Ordered sets capture a wide range of natural concepts (“better than”, “before than”, “greater than” ...), so much effort has gone into their study. Among other things, several drawing techniques for ordered sets have been developed over the years. The most common scheme consistently used to represent ordered sets is known as “upward drawing”, or Hasse diagram. It is a graph whose vertices correspond to elements of the set and whose edges correspond to pairs of elements in the

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relation. Because the relation is reflexive, every element will have an edge to itself. In a Hasse diagram, these edges are implied and not drawn. In addition, many other edges are implied by transitivity, so these non essential edges are disregarded as well. Finally, it is always possible to orient such a directed covering graph in such a way that all arrows point upward. A Hasse diagram follows such an upward orientation, so the actual arrows become implied and are not drawn either. Orders are drawn bottom-up: if an element x is smaller than an element y then there exists a path from x to y that is directed upward. See Figure 1 (a) for an example.

Another way to draw an ordered set is a *book embedding*. In the book embedding of an ordered set P, the elements of P are embedded along the spine of the book to form a linear extension, that is, a total ordering of the elements of P that is consistent with the original ordering. The edges are then drawn on different *pages* of the book, in such a way that edges drawn in the same page do not intersect. Similarly to Hasse diagrams, in a book embedding only essential edges are drawn (that is, edges that are not implied by reflexivity or transitivity). Figure 1 (b) provides an example of a book embedding for the order shown on Figure 1 (a). The *pagenumber* of P is the minimum number of pages needed to draw a book embedding of P.

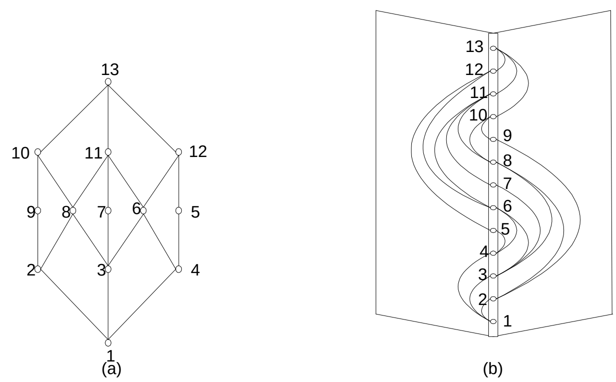

FIGURE 1. An upward drawing of an ordered set(a) and an optimal book embedding (b) of the same ordered set.

A large number of problems in different domains can be formulated as graph layout problems (see Diaz et al. [3] for a survey). Bernhart and Kainen [2] were first to study book embedding for *graphs*. A book embedding (or stack layout) of a graph G consists of an embedding of its nodes along the spine of a book and embeddings of its edges on pages so that edges embedded on the same page do not intersect. The *pagenumber* of G, $page(G)$ (sometimes referred to as *stacknumber* in the literature), is the minimum number of pages needed, taken over all permutations on the vertices of G. Applications of stack layouts of graphs include sorting permutations, fault tolerant VLSI design, complexity theory, compact graph encodings, compact routing tables, and graph drawing. The idea of book embedding and page number was first adapted to ordered sets by Nowakowski and Parker [7]. As already suggested, the pagenumber of an ordered set P is the pagenumber of P's Hasse diagram viewed as a directed graph. The topic of book embedding of ordered sets as turned out to be a difficult one, with very few results so far. In fact, most of the known results relate to classes of ordered sets with a pagenumber two, but even the question of a general characterization of ordered sets with pagenumber two is still open. On the other hand, several questions have been shown to be NP-complete: first and foremost, computing the page number of a general ordered set is NP-complete. In fact, the question remains NP-complete even if the order of the nodes along the spine is fixed. Also NP-complete is the question of knowing if a general order can be embedded into six pages, or simply computing the pagenumber of a bipartite order. Refer to [6] for an extensive review of these results.

Perhaps the only challenging class of ordered sets for which a precise solution is known is the class of series-parallel planar ordered sets: Alzohairi and Rival [1] showed that the pagenumber of any series-parallel planar ordered set is at most two. In a related result, Di Giacomo et al. [5] provided a linear time algorithm to embed series-parallel lattices into two pages. Another known result was provided by Syslo in [9], for complete bi- and tri-partite ordered sets: a complete bipartite ordered set having n_1 minimal elements and n_2 maximal elements has a pagenumber of $min\{n_1, n_2\}$. For complete tripartite ordered sets having n_1 elements of height zero, n_2 elements of height one and n_3 elements of height two, the pagenumber is $page(P) = min\{n_2, n_1 + n_3\}$.

In this paper, we look at the question of the pagenumber for some restricted classes of ordered sets. We first prove that the pagenumber of a bipartite interval order P is equal to the maximum pagenumber of a complete suborder of P. We use a technique that relies on easily identifying complete suborders within a given bipartite interval order. This yields to a polynomial time algorithm for finding the pagenumber of bipartite interval orders. All of these results are covered in Section 3. Then, in Section 4, we use the same strategy to establish an upper bound for the pagenumber of bipartite ordered sets. We also give an upper bound for the pagenumber of complete multipartite ordered sets in Section 5. Finally we discuss the effect of a number of diagram operations on the pagenumber of ordered sets and we list several open questions we consider worth investigating in Section 6.

We first begin with some general definitions provided in Section 2.

2. Definitions

Definition 1 (Ordered sets)**.** *An* ordered set *(or simply* order*)* $(P, \leq)$ *is a reflexive, antisymmetric and transitive binary relation* $\leq$ *over a set* P*:*

- $\forall x \in P, x \leq x$ *(reflexivity),*
- $\forall x, y \in P, x \leq y$ *and* $y \leq x \Rightarrow x = y$ *(antisymmetry),*
- $\forall x, y, z \in P, x \leq y$ *and* $y \leq z \Rightarrow x \leq z$ *(transitivity).*

Two elements x *and* y *of* P *are* comparable *if either* $x \leq y$ *or* $y \leq x$*. If* x *and* y *are not comparable, then they are* incomparable*. We note* $<$ *the* strict *relation corresponding to* $\leq$*: for all* x, y *in* P*,* $x < y$ *if and only if* $x \leq y$ *and* $x \neq y$*. We say that* y *is an* upper cover *of* x*, or equivalently that* x *is* a lower cover *of* y*, denoted* $x \prec y$*, if* $x < y$ *and there is no element* z *in* P *such that* $x < z < y$*. We call* covering relations *the subset of the ordering relation restricted to the covers. The set of* successors *(resp.* predecessors*) of* x *in* P*, denoted Succ(x) (resp. Pred(x)), is the set of all elements* y *in* P *such that* $x \leq y$ *(resp.* $x \geq y$*). If Succ(x)=* $\emptyset$ *(resp. Pred(x)=* $\emptyset$ *), we say that* x *is* maximal *(resp.* minimal*) in the order.*

Definition 2 (Chains, height, sub-order, (complete) bipartite orders, multipartite orders, linear extensions)**.** *A* chain *of an ordered set* $(P, \leq)$ *is a set of pairwise comparable elements of* P*. If the order itself is a chain, then we say that* $(P, \leq)$ *is a* total order*. The* height *of an order is the length of its longest chain (number of elements in the chain minus one). The height of an element* x *of the order is the maximum length of a chain joining a minimal element of the order to* x*. An order is* bipartite *if its height is one (that is, it doesn't have a chain of more than two elements, or again every element is either minimal, or maximal or both). An order is* multipartite *if for all* x, y *in* P*,* $x \prec y$ *implies that the height of* y *is equal to the height of* x *plus one. A bipartite order is* complete *if for all* x, y *in* P*, if* x *is minimal and* y *is maximal then* $x \leq y$*. An order* $(P', \leq_{P'})$ *is a* sub-order *of an order* $(P, \leq)$ *if* $P' \subset P$ *and for all* x, y *in* P'*,* $y \leq_{P'} x \Rightarrow y \leq x$*. Finally, a* linear extension $\leq_l$ *of an order* $(P, \leq)$ *is a relation such that* $(P, \leq_l)$ *is a total order and* $(P, \leq)$ *is a suborder of* $(P, \leq_l)$*.*

Definition 3 (Book embedding, Page number)**.** *Let* $L(P)$ *denote the set of all linear extensions of an ordered set* $(P, \leq)$*. A* book embedding *of* P *with respect to* $L \in L(P)$ *is the embedding of the Hasse Diagram of* P*, with its vertices placed on the spine with respect to* L *(that is, vertices of* P *are drawn on the spine of the book in the order defined by* L*, and edges drawn on the same page do not intersect). The* pagenumber *of* P *with respect to* L*,* $\text{page}(P, L)$*, is the smallest number of pages* k *such that* P *has a book embedding on* k *pages. The* pagenumber *of* P*,* $\text{page}(P)$*, is* *the minimal number of page required over all linear extensions:* page$(P)=$ *min*$\{$page$(P, L)\ :\ L \in L(P)\}$.

Definition 4 (Interval orders)**.** *An interval representation of an ordered set* $(P, \leq)$ *is a function that assigns to each element* x *of* P *an interval* I_x *on the real line such that for each elements* x, y *of* P*,* $x < y$ *if and only if every points of* I_x *are less than every point of* I_y*. If an ordered set* $(P, \leq)$ *has an interval representation, then we call* $(P, \leq)$ *an* interval order*.*

Figure 2 shows an example of an interval order. Interval orders have a very simple characterizations [4]: an ordered set P is an interval order if and only if it does not contain a subset $\{u, v, x, y\}$ of P such that $u < v$ and $x < y$ are the only comparabilities among these elements. In any interval order P the following important condition also holds: the sets of predecessors (as well as the sets of successors) are linearly ordered with respect to inclusion. That is, for all $x, y \in P$, either *Pred(x)* $\subseteq$ *Pred(y)* or *Pred(x)*$\supseteq$ *Pred(y)*.

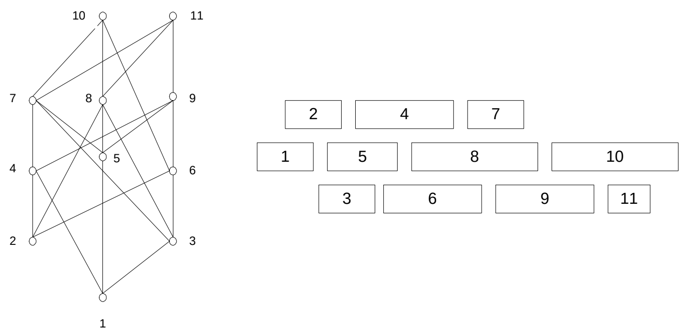

FIGURE 2. An interval order its interval representation.

3. The Pagenumber of Bipartite Interval Orders

In this section, we explore the question of the pagenumber of bipartite interval orders. We give an exact solution, which can be computed in polynomial time. We show that our result does not extend to multipartite interval orders.

3.1. **Main Result.**

In this section, we explore the question of the pagenumber of bipartite interval orders.

Theorem 5. *The pagenumber of a bipartite interval order is equal to the maximum pagenumber of complete suborders of* P*.*

Note first that if an ordered set $(P', \leq_{P'})$ is a suborder of a bipartite order $(P, \leq)$, then $page(P') \leq page(P)$. Indeed, any book embedding of P can be used as a book embedding of P'. Note that this relation between the pagenumber of an order and the pagenumber of its suborders does not hold in general, as we will see in Section 6.

In order to prove Theorem 5, we first introduce two lemmas. Let $(P, \leq)$ be a bipartite interval order. Let $M = (m_1, m_2, \ldots, m_m)$ be the list of minimal elements of $(P, \leq)$ arranged in a decreasing order with respect to the inclusion relation of the sets of successors, i.e. $Succ(m_1) \supseteq Succ(m_2) \supseteq \ldots \supseteq Succ(m_m)$. Let $N = (n_1, n_2, \ldots, n_n)$ be the list of maximal elements of P arranged in decreasing order with respect to the inclusion relation of the sets of predecessors, i.e. $Pred(n_1) \supseteq Pred(n_2) \supseteq \ldots \supseteq Pred(n_n)$. In the following, for simplicity we simply write $P = (M, N)$ for the bipartite ordered set $(P, \leq)$. The following Lemma holds:

Lemma 6. *Let $P = (M, N)$ be a bipartite interval order, and let $P' = (M', N')$ be a complete suborder of P. Then there exists i and j such that $M' \subseteq \{m_1, m_2, \cdots, m_i\}$ and $N' \subseteq \{n_1, n_2, \cdots, n_j\}$ and $(\{m_1, m_2, \cdots, m_i\}, \{n_1, n_2, \cdots, n_j\})$ is a complete bipartite suborder of P.*

Proof. Let j be the largest index such that $n_j \in N \cap N'$. Index j exists necessarily since $N' \subseteq N$. Since $P' = (M', N')$ is a complete suborder of P, $N' \subseteq \{n_1, n_2, \cdots, n_j\}$ and $Pred(n_j) \subseteq Pred(n_k)$ for every $k \leq j$, we have that $(M', \{n_1, n_2, \cdots, n_j\})$ is a complete suborder of P. Likewise, let i be the largest index such that $m_i \in M \cap M'$. Index i exists necessarily since $M' \subseteq M$. Since $P' = (M', N')$ is a complete suborder of P, $M' \subseteq \{m_1, m_2, \cdots, m_i\}$ and $Succ(m_i) \subseteq Succ(m_k)$ for every $k \leq i$, we have that $(\{m_1, m_2, \cdots, m_i\}, N')$ is a complete suborder of P. Thus, $(\{m_1, m_2, \cdots, m_i\}, \{n_1, n_2, \cdots, n_j\})$ is a complete bipartite suborder of P. □

Lemma 6 shows that any complete bipartite suborder of P is also a suborder of another suborder of P with the structure $P(i, j) = (\{m_1, m_2, \cdots, m_i\}, \{n_1, n_2, \cdots, n_j\})$, for some i and j. Recall that the pagenumber of a complete bipartite order is equal to the minimum between the number of minimal element and the number of maximal elements of the order [9]. Thus, in order to find a complete bipartite suborder of P with maximum pagenumber, it is sufficient to look at this structure $P(i, j)$, with $page(P(i, j)) = min\{i, j\}$.

Lemma 7. *Let $P = (M, N)$ be a bipartite interval ordered set, and let $P(i, j)$ be a complete suborder of P with maximum pagenumber. If there exits k and l such that $min(i, j) < k \leq n$ and $min(i, j) < l \leq m$, then n_k is incomparable to m_l in P.*

Proof. Suppose that $j > i$, therefore $page(P(i,j)) = min\{i,j\} = i$. Suppose that $n_k > m_l$ in P for some indexes k and $l > i$. Thus, by definition, $Succ(m_l) \supseteq \{n_1, n_2, \cdots, n_k\}$. Since $Succ(m_l) \subseteq Succ(m_r)$ for every $r \leq l$, and $l > i$, we have that $Succ(m_{i+1}) \supseteq \{n_1, n_2, \cdots, n_k\}$ and therefore $P(i+1,k) = (\{m_1, m_2, \cdots, m_i, m_{i+1}\}, \{n_1, n_2, \cdots, n_k\})$ is a complete suborder of P. But then, since $k > i$, we have $page(P(i+1,k)) = min$ $\{i+1,k\} = i+1 > page(P(i,j))$. This contradicts the choice of $P(i,j)$. The same argument apply if $j < i$. □

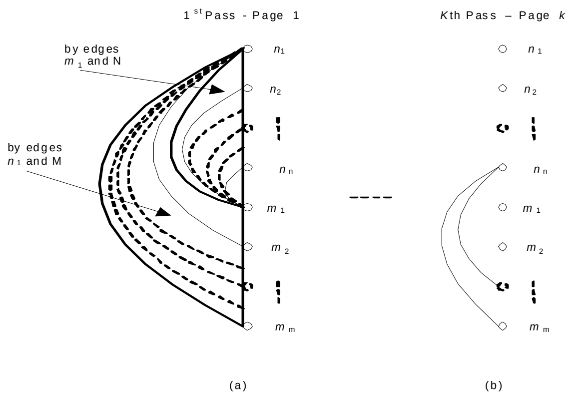

FIGURE 3. An illustration for Theorem 5.

We are now in a position to prove Theorem 5.

Proof (Theorem 5). Consider a complete suborder $P' = P(i,j)$ of P with maximum pagenumber. We prove that the bipartite interval ordered set P can also be embedded in $page(P')$ pages, that is, in min $\{i,j\}$ pages.

For our embedding, we use the following linear extension L of $(P, \leq)$:

$$m_m < m_{m-1} < \cdots < m_1 < n_n < n_{n-1} < \cdots < n_1$$

Without loss of generality, we assume that $i < j$. We proceed as follows: on the first page, we draw all covering relations of m_1 (thus we draw edges between m_1 and n_k for all k), as well as all covering relations of n_1 (thus we draw edges between n_1 and m_l for all l). Clearly, we can fit all these covering relations on the first page, because m_1 is drawn above every other m_l and n_1 is drawn above every other n_k. This is illustrated on Figure 3 (a). After this first page, all relations involving n_1 or m_1 have been drawn. Thus, on the second page, we can now draw all the (remaining) covering relations of n_2 and all the (remaining) covering relation of m_2 in the same way. We continue following the same technique, so that for all $p \leq i$, we draw on the p^{th} page all the (remaining) covering relations of n_p and all the (remaining) covering relation of m_p (see Figure 3 (b)). Thus, once we have drawn the i^{th} page, we have drawn all the covering relations of $n_1, n_2, \cdots, n_i, m_1, m_2, \cdots, m_i$. But Lemma 7 guarantees that this is actually all the covering relations of P, so we have a book embedding of $(P, \leq)$ in $i =$*page*(P') pages. $\square$

This result is illustrated in Figure 4 and 5. Figure 4 shows a bipartite interval order. Its complete suborder with the largest pagenumber is $P'(5, 4)$, which can can be embedded in four pages. Thus, a four pages book embedding of the order is constructed with the linear extension L : $b < c < e < a < d < n < m < l < g < f < k < j < i < h$, as shown in Figure 5.

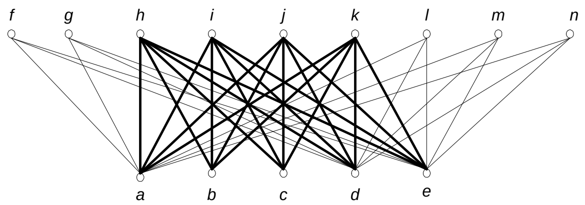

FIGURE 4. A bipartite interval order.

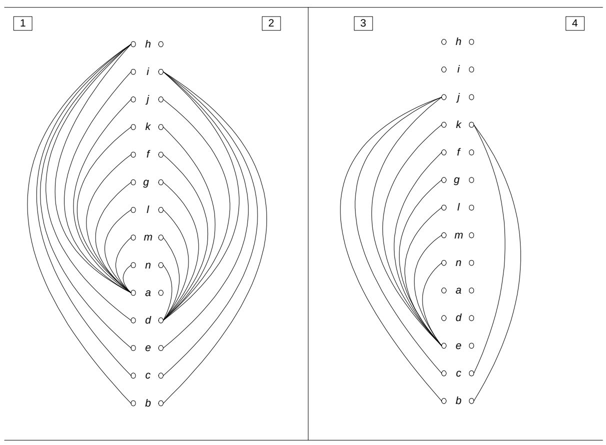

FIGURE 5. Four pages book embedding of the order of Figure 4.

3.2. **A Polynomial Time Algorithm.** The proof of Theorem 5 was a constructive one, so we can infer an algorithm from it. Algorithm 1 receives a bipartite interval order as input, along with the list of minimal elements sorted in a decreasing order with respect to the inclusion relation of the sets of successors and the list of maximal elements sorted in a decreasing order with respect to the inclusion relation of the sets of predecessors. It produces as output an optimal book embedding of the input order.

The drawing algorithm has two parts: the preprocessing (sorting the list of minimal elements of P in a decreasing order with respect to the inclusion relation of the sets of successors and sorting the list of maximal elements of P in decreasing order with respect to the inclusion relation of the sets of predecessors), which is not detailed here, and drawing stage, which is provided. It is easy to see that both stages of the algorithm can be performed within a time complexity of $O(n^2)$ where n is the number of elements in P.

Algorithm 1 DrawBipartiteIntervalOrder(IN Order P, VertexArray Min(), VertexArray Max())

Input = A bipartite interval order P
Input = An array of minimum elements of P Min = $\{m_1, m_2, \ldots, m_m\}$, sorted by decreasing set of successors
Input = An array of maximum elements of P Max = $\{n_1, n_2, \ldots, n_n\}$, sorted by decreasing set of predecessors
Output = an optimal embedding of P
k=0;pageNumber=0;
Draw $m_m < m_{m-1} < \ldots < m_1 < n_n < n_{n-1} < \ldots < n_1$ on the spine
while $k \leq m$ AND $k \leq n$ AND $m_k \prec n_k$ AND $m_k \prec n_k$ is not drawn **do**
  increment pageNumber
  while $\exists e \in \{n_k, n_{k+1}, \ldots, n_n\}$ such that $m_k \prec e$ and $m_k \prec e$ is not drawn **do**
    Draw edge $m_k \prec e$ on page pageNumber
    Mark edge $m_k \prec e$ as drawn
  end while
  while $\exists e \in \{m_k, m_{k+1}, \ldots, m_m\}$ such that $e \prec n_k$ and $e \prec n_k$ is not drawn **do**
    Draw edge $e \prec n_k$ on page pageNumber
    Mark edge $e \prec n_k$ as drawn
  end while
  increment k;
end while

4. An Upper Bound for the Pagenumber of Bipartite Ordered Sets

The strategy used in Section 3 for creating an optimal book embedding of a bipartite interval order can actually be adapted to create a (non necessarily optimal) book embedding of any any bipartite order, this providing us with an upper bond for this class of orders.

We introduce the concept of *zig-zag* inside a bipartite order:

Definition 8 (Zig-zag)**.** *Let $P = (M, N)$ be a bipartite ordered set. A* zig-zag *Z of length $2l$ in P is a partition of M into l sets $M_1, M_2, \cdots, M_l$ and a partition of N into l sets $N_1, N_2, \cdots, N_n$ such that*

$\mathrm{Succ}(M_i) \subseteq N_i \cup N_{i+1}$ *for* $1 \leq i < l$ *and* $\mathrm{Succ}(M_l) \subseteq N_l$

We are going to use zig-zags of length four in bipartite order, as illustrated in Figure 6. It is worth noting that any non complete bipartite order $P = (M, N)$ has at least one zig zag of length four: it is enough to take $x \in N$ and $y \in M$ such that x and y are incomparable, and then $N_1 = \{x\}, N_2 = N \setminus \{x\}, M_1 = M \setminus \{y\}$, and $M_2 = \{y\}$ defines a zig-zag of length four. We can extend this definition to complete bipartite by having $N_1 = \emptyset$ or $M_2 = \emptyset$, but since an optimal solution to the question of book embedding of complete bipartite orders is already known ([9]), such an extension is not necessary.

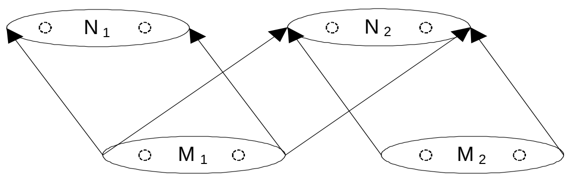

Figure 6. A zig-zag of length four.

Theorem 9 provides an upper bound to the pagenumber of any non complete bipartite order by using a *zig-zag* cover of length four:

Theorem 9. *Let $P = (M, N)$ be a non complete, bipartite ordered set. Let $Z = M_1, M_2, N_1, N_2$ be a* zig-zag *of length 4 that covers P. We have*

page(P)=max $\{|M_1|, |N_2|\}$.

Proof. Assume that $M_1 = \{m_{1,1}, m_{1,2}, \cdots, m_{1,m}\}$, $M_2 = \{m_{2,1}, m_{2,2}, \cdots, m_{2,n}\}$, $N_1 = \{n_{1,1}, n_{1,2}, \cdots, n_{1,p}\}$ and $N_2 = \{n_{2,1}, n_{2,2}, \cdots, n_{2,q}\}$. We prove the Theorem by constructing a book embedding of P.

Let L be a linear extension of P obtained by enumerating the elements of M_1, then the elements of M_2, then the elements of N_2 and then the elements of N_1, in the following order:

$$m_{1,1} < m_{1,2} < \cdots < m_{1,m} < m_{2,1} < m_{2,2} < \cdots < m_{2,n} <$$
$$n_{2,1} < n_{2,2} < \cdots < n_{2,q} < n_{1,1} < n_{1,2} < \cdots < n_{1,p}$$

We use the ordering of L to put the elements along the spine of the book. On the first page of the embedding, we draw all the covering relations of $m_{1,1}$ and all the covering relation of $n_{2,1}$. We can do it because $n_{2,1}$ is the lowest of the elements of N on the spine, and $m_{1,1}$ is the lowest of all elements, so the covering relations of $m_{1,1}$ will reach $n_{2,1}$ and higher, leaving space between $n_{2,1}$ and every other elements of M. Likewise, on page two, we can draw all the (remaining) covering relation of $m_{1,2}$ if this element exist, and all the (remaining) covering relations of $n_{2,1}$ if this element exists. Continuing with the same principle, after $k = max\{m, q\}$ pages, we will have drawn all covering relations of all elements of M_1 as well as all covering relation of all elements of N_2. Since there are no relations between elements of N_1 and elements of M_2, we have in fact drawn all the relations of P, so we have a book embedding of P in $max\ \{|M_1|, |N_2|\}$ pages. □

5. Pagenumber of Complete Multipartite Ordered Sets

In this Section, we give an upper bound for the pagenumber of arbitrary complete multipartite ordered sets. We consider an arbitrary, complete multipartite ordered set E of height $k-1$, that is, an order that has k levels and such that every element at one level has covering relations with every other elements at the preceding level and with every other elements at the following level (and obviously none other). For simplicity, we note n_1 the set of elements of the first level (with height zero), n_2 the set of elements at the second level (with height one), and so on until n_k which is the set of the maximal elements, at the level k (with height $k-1$). So for all x, y, height(x) = height(y)-1 if and only if $x \prec y$.

We are going to use a particular type of book embedding, which we call *directed*. Directed embedding have two properties: first, when a covering relation $x \prec y$ between two elements x and y is drawn on a page, then either every other upper covering relations of x are also drawn on that same page, or every other lower covering relations of y are also drawn on that same page. Second, the "direction" used to draw covering relations between the level of x and the succeeding level of y is the same for every other element between the two levels, either going "up" from the level of x to the level of y (every upper covering relations) or going "down" from the level of y to the level of x (every lower covering relations). Formally:

Definition 10 (Directed Embedding of Complete Multipartite Ordered Sets)**.** *A book embedding of a complete, multipartite ordered set E of height $k-1$ is* directed *if,*

$\forall i \in [1, k-1]$*, either*

$\forall x \in n_i$*, all upper covering relations of x are drawn on the same page (case*

one),
or
$\forall y \in n_{i+1}$*, all lower covering relations of* y *are drawn on the same page (case two).*

If the covering relations between the level $k-1$ *and the last level* k *are drawn using case one, we say that the book embedding of* E *is* up*, which we write* $E^{\uparrow}$*. Otherwise (case two), we say that the book embedding of* E *is* down*, which we write* $E^{\downarrow}$*. We denote by* $|E|$ *the number of pages used by the directed book embedding of* E*.*

We provide an upper bound for the pagenumber of arbitrary complete multipartite ordered sets E by constructing a directed embedding of it. We construct this embedding recursively, by extending a directed embedding of the first $i-1$ levels of E to a directed embedding of the first i levels of E. By convention, we write $E_i^{\downarrow}$ (resp. $E_i^{\uparrow}$) for a directed down (resp. up) embedding of the first i levels of the order.

Theorem 11. *Let* $E = (n_1, n_2, \ldots, n_k)$ *be an arbitrary complete multipartite ordered set of height* $k-1 \geq 2$*. The pagenumber of* P *is bounded by*

$$|E_k| = min\{|E_k^{\uparrow}|, |E_k^{\downarrow}|\}$$

where

$$|E_2^{\downarrow}| = |n_2| \qquad |E_2^{\uparrow}| = |n_1| \tag{12}$$

$$|E_3^{\downarrow}| = |n_1| + |n_3| \qquad |E_3^{\uparrow}| = |n_2| \tag{13}$$

And for all $i \in [4, \ldots, k]$

$$|E_i^{\downarrow}| = max\{|E_{i-2}^{\downarrow}|, |n_i| + |n_{i-2}|\} \tag{14}$$

$$|E_i^{\uparrow}| = min\{|E_{i-1}^{\downarrow}|, max\{|E_{i-1}^{\uparrow}|, |n_{i-1}| + |n_{i-2}| - 1\}\} \tag{15}$$

.

A direct consequence of Theorem 11 is that the pagenumber of any complete multipartite ordered set of height three is bounded by $min\{|n_1| + |n_3|, |n_2| + |n_4|, |n_2| + |n_3| - 1\}$. Figure 7 (a) shows such an example, with a directed embedding $E_4^{\uparrow}$ where by $|E_4^{\uparrow}| = |n_2| + |n_3| - 1 = 4$.

We now prove Theorem 11:

Proof (Theorem 11). The values for $|E_2^{\downarrow}|, |E_2^{\uparrow}|, |E_3^{\downarrow}|$ and $|E_3^{\uparrow}|$ are direct consequence of [9].

To prove (14), note that by definition, $E_{i-2}^{\downarrow}$ is an embedding on which $|n_{i-2}|$ pages have edges between one element of n_{i-2} and every elements of n_{i-3}, while the other $|E_{i-2}^{\downarrow}| - |n_{i-2}|$ pages have no edges between n_{i-2} and n_{i-3}. We can extend the embedding $E_{i-2}^{\downarrow}$ in the following way: on each page that contains edges between one element $x \in n_{i-2}$ and the elements

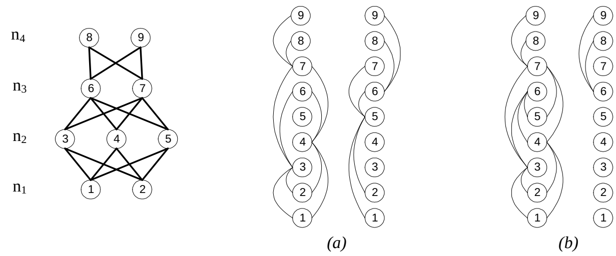

FIGURE 7. A complete multipartite order and a four (a) and three (b) pages book embedding.

of n_{i-3}, we also add edges between x and every elements of n_{i-1}. This is possible because no other elements of n_{i-2} have edges drawn on this page in $E^{\downarrow}_{i-2}$. This approach will ensure that every edge between n_{i-2} and n_{i-1} will be drawn on $|n_{i-2}|$ existing pages of $E^{\downarrow}_{i-2}$, so we only need to draw the edges between elements of n_{i-1} and elements of n_i. For this, with have $|E^{\downarrow}_{i-2}| - |n_{i-2}|$ pages remaining in $E^{\downarrow}_{i-2}$. On each page of $E^{\downarrow}_{i-2}$ that has no edges between n_{i-2} and n_{i-3}, we add the edges between one element of n_i and all elements of n_{i-1} until we either run out of elements of n_i or we run out of pages in $E^{\downarrow}_{i-2}$. In the former case, we were able to complete the embedding $E^{\downarrow}_i$ using the existing pages of $E^{\downarrow}_{i-2}$, obtaining an embedding $E^{\downarrow}_i$ such that $|E^{\downarrow}_i| = |E^{\downarrow}_{i-2}|$. In the latter case, after we run out of existing pages we start adding new pages to finish the elements of n_i. We have used $|n_{i-2}|$ pages before starting the elements of n_i, and exactly $|n_i|$ pages for the elements of n_i, thus creating an embedding $E^{\downarrow}_i$ having $|n_i| + |n_{i-2}|$ pages. This proves (14).

To prove (15), we first note that $E^{\downarrow}_{i-1}$ can always be extended so that in each page containing edges between one element $x \in n_{i-1}$ and the elements of n_{i-2}, we also add edges between x and every elements of n_i. This gives a solution for $E^{\uparrow}_i$ having $|E^{\downarrow}_{i-1}|$ pages. In addition, by definition, $E^{\uparrow}_{i-1}$ is an embedding on which $|n_{i-2}|$ pages have edges between one element of n_{i-2} and every elements of n_{i-1}, while the other $|E^{\uparrow}_{i-1}| - |n_{i-2}|$ pages have no edges between n_{i-2} and n_{i-1}. We can extend the embedding $E^{\uparrow}_{i-1}$ in the following way: on the first page that contains an edge between one element $x \in n_{i-2}$ and the elements of n_{i-1}, we also add edges between the *last* element of n_{i-1} in the linear extension and every element of n_i. This is always possible because there are no edge between elements of n_{i-1} and elements of n_i in $E^{\uparrow}_{i-1}$. Then, on each page that has no edges between n_{i-2} and n_{i-1}, we add the edges between one element of n_{i-1} (except the last one in the linear extension) and all elements of n_i until we either run out of elements of n_{i-1} or we run out of pages. In the former case, we were able to complete the embedding $E_i^{\uparrow}$ using the existing pages of $E_{i-1}^{\uparrow}$, obtaining an embedding $E_i^{\uparrow}$ such that $|E_i^{\uparrow}| = |E_{i-1}^{\uparrow}|$. In the latter case, after we run out of existing pages we start adding new pages to finish the elements of n_{i-1}. We have used $|n_{i-2}|$ pages on which we have drawn the edges of only one elements of n_{i-1}, so we need exactly $|n_{i-1}| - 1$ pages for the remaining elements of n_{i-1}, thus creating an embedding $E_i^{\uparrow}$ having $max\{|E_{i-1}^{\uparrow}|, |n_{i-1}| + |n_{i-2}| - 1\}$ pages. This proves (15).

Since both $E_i^{\downarrow}$ and $E_i^{\uparrow}$ are valid embedding, the Theorem is proved. □

It should be noted that this upper bound is not tight: Figure 7 (b) shows a three pages book embedding of the multipartite order of length four for which Theorem 11 gives an upper bound of four. We however argue that this results provides in general a "good" upper bound in the sense that it provides a solution that is computed on a "sliding window" of only four levels. In other word, the solution is not necessarily optimal, but it does not keep degrading from level to level.

6. Other Properties of the Pagenumber and Open Problems

In this Section, we review some open questions that we believe are worth investigating.

One question of interest is the impact of removing an element of an ordered set on the pagenumber of the resulting sub-order. This kind of removal operation is typically useful when using recursive proofs. We write $P \setminus x$ the suborder of $(P, \leq)$ obtained by removing the element x from the set P. In general, it is easy to find examples where $page(P \setminus x) < page(P)$ as well as examples where $page(P \setminus x) > page(P)$. Figure 8 shows examples going both ways. In fact, the difference between $page(P \setminus x)$ and $page(P)$ could be arbitrarily large, as illustrated on Figure 8 (b): the complete tri-partite order $K(n, 1, n)$ has a pagenumber one, while it suborder the complete bipartite order $K(n, n)$ has a pagenumber n.

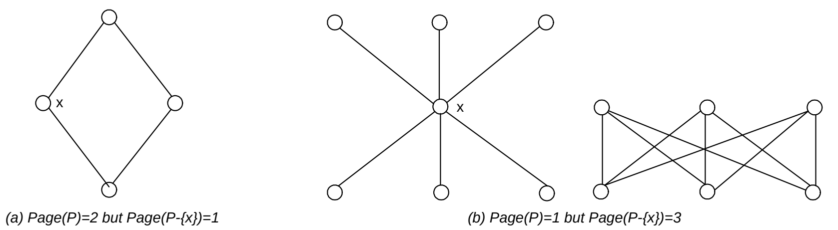

Figure 8. Removing an element from the order can decrease the pagenumber (a) or increase it (b)

Let $(P, \leq)$ be an ordered set and let x be an element in P. We say that x is *irreducible* if x has a unique lower cover and a unique upper cover in P. If we remove an irreducible elements from an ordered set, the following is easy to show:

Lemma 16. *Let P be an ordered set and let x be an irreducible element such that $u \prec x \prec v$.*

(a) If $u \prec x \prec v$ is the unique chain from u to v then $\text{page}(P) \leq \text{page}(P \setminus x)$.

(b) If $u \prec x \prec v$ is not the unique chain from u to v then $\text{page}(P) \geq \text{page}(P \setminus x)$.

Another operation is *subdivision*. We subdivide a covering edge $u \prec v$ in P by adding a new element x between u and v and creating the new relations $u \prec x$ and $x \prec v$ in P. Clearly, we obtain a new ordered set, in which x is an irreducible element. From a drawing viewpoint, it is obvious that edge subdivision does not change the planarity (or non planarity) of an ordered set. With respect to book embedding drawings, Lemma 16 shows that subdividing an edge does not increase the pagenumber of the resulting order. However, can it be decreased?

Open Question 1. *Let P' be the ordered set obtained from P by subdividing a covering relation. Does* $\text{page}(P) = \text{page}(P')$*?*

Another class of ordered sets which is of general interest is the class of N-free ordered sets. An ordered set is N-free if its diagram contains no sub-diagram isomorphic to N, that is, four distinct elements a, b, c and d such that $a \prec c$, $b \prec d$ and $b \prec c$ are the only comparabilities among these elements (these relationships look like the letter N, hence the name). This class is larger class than the class of series parallel orders. How difficult is the computation of the pagenumber of N-free ordered sets is unknown, including in the case of planar N-free ordered sets.

One of the reasons why N-free orders are important is because every finite ordered set can be embedded into an N-free ordered set, in polynomial time. A simple way to achieve this is to just subdivide every covering relations of the ordered set. A more efficient way (in term of subdivisions) to obtain the same result is to subdivide only the covering relations that correspond to the diagonal of an N, and repeat the operation once if necessary [8]. This shows that a positive answer to Open Question 1 will imply a positive answer to the following one:

Open Question 2. *Is the pagenumber problem as difficult for N-free ordered sets as it is for general ordered sets?*

We already pointed out that removing an element from an ordered set can increase the pagenumber arbitrarily. However, it is easy to see that it cannot decrease the pagenumber by more than one. We define an ordered set P as a *pagenumber k-critical*, or simply *k-page-critical* if $page(P) = k$ and the removal of any element of P reduces its pagenumber. That is, $page(P \setminus x) < page(P)$ for any x in P.

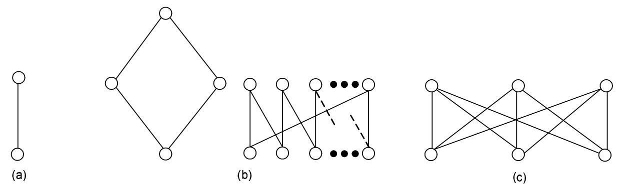

FIGURE 9. Page-critical ordered sets with pagenumber 1, 2 and 3.

Thanks to Theorem 5, we know that the only *n-page-critical* bipartite interval orders are the complete bipartite ordered sets $K_{n,n}$. Obviously, the only page-critical ordered set P with $page(P) = 0$ is the singleton. It is also easy to check that the only *1-page-critical* ordered set is that shown in Figure 9 (a), and the only *2-page-critical* orders are of the form shown Figure 9 (b). Figure 9 (c) gives and example of a *3-page-critical* ordered set. However, for larger k, the question of characterization is still open:

Open Question 3. *Characterize the ordered sets which are k-page-critical.*

In most instances of optimization problems, transformation techniques are developed to transform an optimal solution into some other optimal solution with a desired structure. When it comes to book embedding, the effect of simple transformations is often unknown. For example, if an optimal book embedding is provided for a given linear extension of the order, does it help finding an optimal book embedding for another linear extension of the same order?

Open Question 4. *What is the effect of switching a pair of consecutive incomparable elements in the linear extension on the pagenumber? In general characterize the maximal elements that could be on the top of a linear extension to obtain an optimal book embedding of the order.*

One of the best known operations on diagrams of ordered sets is the *pushdown*. Given an ordered set $(P, \leq)$ and an arbitrary maximal a element of P, the order $P_{pd(a)}$ is obtained from $(P, \leq)$ by "pushing down" a, that is, by transforming $(P, \leq)$ such that a becomes a minimal element and all lower covers of a become upper covers of a (that is, $x \prec a$ in P if and only if $a \prec x$ in $P_{pd(a)}$, and a is a minimal element of $P_{pd(a)}$).

Nowakowski and Parker asked whether the pagenumber of an order was preserved by pushdown [7]. We show that it is not the case, with an example

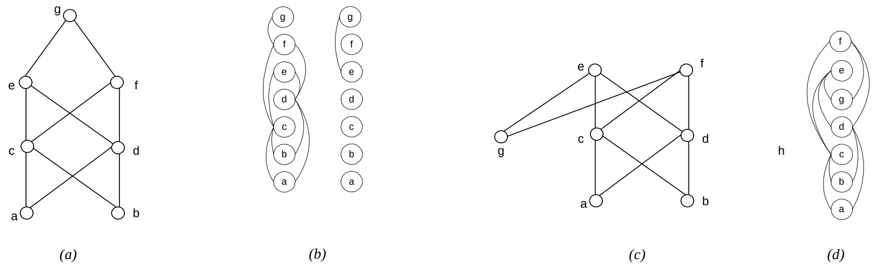

FIGURE 10. The pushdown operation for g lowers the pagenumber from three to two.

for which a pushdown operation lowers a pagenumber from three to two (see Figure 10). The order P shown in (a) has a pagenumber of three. Indeed, (b) shows an embeeding on three pages. Moreover, the embedding cannot be done on two pages for the following reason: since P is a complete multipartie order, all of its linear extensions have the same number of pages. Thus we may restrict our analysis to one linear extension, say $L : a < b < c < d < e < f < g$. Let's assume that there is a book embedding of P with respect to L with only two pages. The edges $(a \prec c)$ and $(b \prec d)$ must be drawn on two separate pages, say *page 1* and *page 2* respectively. Thus the edges $(c \prec e)$ and $(c \prec f)$ must be on *page 1* and therefore the edge $(d \prec f)$ must be on *page 2*. None of the two pages could contain the edge $(e \prec g)$, a contradiction. Figure 10 (c) shows the same order, transformed with a pushdown on g. As shown in (d), the pagenumber is now two.

We in turn ask a more general question:

Open Question 5. *Is there a relationship between the pagenumber of an ordered set P and the pagenumber of the order $P_{pd(a)}$ obtained from P by a pushdown operation on an element $a \in P$?*

Finally, our last open question relates to the generalization of Theorem 5 to the case of multipartite interval orders.

Open Question 6. *Can Theorem 5 be generalized to the case of tripartite interval orders, and more generaly to the case of k-partite interval orders for $k \geq 2$? Is it at least possible to apply the bound of Theorem 11 to multipartite interval orders?*

7. CONCLUSION

In this paper, we look at the problem of the pagenumber of bipartite ordered sets. We give a polynomial time algorithm finding the exact pagenumber of bipartite interval orders. We also give an upper bound for the pagenumber of general bipartite orders, and of complete multipartite orders. Much work remains to be done on the question of pagenumber, for which few results are known. We provide a series of open questions which we believe are worthwhile investigating.

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