

Solution Devoir 4

Question 1

$$\begin{aligned}
 (a) \quad E[X(t)] &= E[X \cos 2\pi f_0 t + Y \sin 2\pi f_0 t] \\
 &= E[X \cos 2\pi f_0 t] + E[Y \sin 2\pi f_0 t] \\
 &= E[X] \cos 2\pi f_0 t + E[Y] \sin 2\pi f_0 t \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad R_{xy}(t_1, t_2) &= E[x(t_1)x(t_2)] \\
 &= E((X \cos 2\pi f_0 t_1 + Y \sin 2\pi f_0 t_1)(X \cos 2\pi f_0 t_2 + Y \sin 2\pi f_0 t_2)) \\
 &= E(X^2 \cos^2 2\pi f_0 t_1 \cos 2\pi f_0 t_2) \\
 &\quad + E(XY \cos 2\pi f_0 t_1 \sin 2\pi f_0 t_2) \\
 &\quad + E(XY \sin 2\pi f_0 t_1 \cos 2\pi f_0 t_2) \\
 &\quad + E(Y^2 \sin^2 2\pi f_0 t_1 \sin 2\pi f_0 t_2) \\
 &= E(X^2) \cos 2\pi f_0 t_1 \cos 2\pi f_0 t_2 \\
 &\quad + E(XY) \cos 2\pi f_0 t_1 \sin 2\pi f_0 t_2 \rightarrow E(XY) = 0 \\
 &\quad + E(XY) \sin 2\pi f_0 t_1 \cos 2\pi f_0 t_2 \rightarrow E(XY) = 0 \\
 &\quad + E(Y^2) \sin 2\pi f_0 t_1 \sin 2\pi f_0 t_2 \\
 &= \sigma^2 \cos 2\pi f_0 t_1 \cos 2\pi f_0 t_2 \\
 &\quad + \sigma^2 \sin 2\pi f_0 t_1 \sin 2\pi f_0 t_2 \\
 &= \frac{\sigma^2}{2} \cos 2\pi f_0 t_1 (-t_2) + \frac{\sigma^2}{2} \cos 2\pi f_0 t_1 (t_1 + t_2) \\
 &\quad + \frac{\sigma^2}{2} \cos 2\pi f_0 t_1 (t_1 - t_2) - \frac{\sigma^2}{2} \cos 2\pi f_0 t_1 (t_1 + t_2) \\
 &= \sigma^2 \cos 2\pi f_0 t_1 (-t_2) = \sigma^2 \cos 2\pi f_0 \tau \\
 &\quad \text{ou } \underline{\tau = t_1 - t_2}
 \end{aligned}$$

(c) On peut démontrer que $E[X^2(t)] = \sigma^2$
 Since $X(t)$ est stationnaire & large sans

$$\text{et } S_x(f) = \mathcal{F}\{R_x(\tau)\} = \frac{\sigma^2}{2} \delta(f - f_0) + \frac{\sigma^2}{2} \delta(f + f_0)$$

Question 2

$$(a) h(t) = \delta(t - \Delta t)$$

$$H(f) = e^{-j2\pi f \Delta t}$$

$$|H(f)|^2 = 1$$

$$S_y(f) = S_x(f) |H(f)|^2 = S_x(f)$$

$$\text{Donc } R_y(\tau) = R_x(\tau)$$

$$R_{xy}(\tau) = E[X(t)Y(t+\tau)]$$

$$= E[X(t)X(t - \Delta t + \tau)]$$

$$= R_x(\tau - \Delta t)$$

$$\text{aussi } R_{xy}(\tau) = R_x(\tau) * h(\tau)$$

$$= R_x(\tau) * \underline{\delta(\tau - \Delta t)}$$

$$= \underline{\underline{R_x(\tau - \Delta t)}}$$

$$(b) h(t) = e^{-3t} u(t)$$

$$H(f) = \frac{1}{3 + j2\pi f}$$

$$|H(f)|^2 = \frac{1}{9 + (2\pi f)^2}$$

$$S_y(f) = S_x(f) |H(f)|^2$$

$$= \underline{\underline{\frac{S_x(f)}{9 + (2\pi f)^2}}}$$

$$R_y(\tau) = \frac{1}{6} R_x(\tau) * e^{-3|\tau|}$$

$$R_{xy}(\tau) = R_x(\tau) * h(\tau) \\ = R_x(\tau) * e^{-\frac{1}{3}\tau} u(\tau)$$

$$(c) \quad y(t) = \int_{t-T}^{t+T} x(t') dt' \\ = x(t) * \pi\left(\frac{1}{2T}\right)$$

$$S_y(f) = S_x(f) |H(f)|^2$$

$$H(f) = 2T \sin(2\pi f)$$

$$|H(f)|^2 = 4T^2 \sin^2(2\pi f)$$

$$S_y(f) = 4T^2 S_x(f) \sin^2(2\pi f)$$

$$R_y(\tau) = \mathcal{F}\{S_y(f)\} \\ = \underline{R_x(\tau) * A\left(\frac{\pi}{2T}\right)}$$

Question 3

$$\iint_0^4 f_{xy}(x,y) dx dy = 1$$

$$\iint_0^4 A xy dx dy = 1$$

$$\int_0^4 \left(\frac{A}{2} x^2 y \right) \Big|_0^y dy = 1$$

$$\int_0^4 \frac{A}{2} y^3 dy = 1$$

$$\frac{A}{8} y^4 \Big|_0^4 = 1$$

$$\frac{256A}{8} = 1$$

$$32A = 1 \\ A = \underline{\underline{\frac{1}{32}}}$$

$$\begin{aligned}
 f_y(y) &= \int_0^{\infty} f_{xy}(x, y) dx \\
 &= \int_0^y \frac{xy}{32} dx \\
 &= \frac{x^2 y}{64} \Big|_0^y \\
 &= \begin{cases} \frac{y^3}{64} & 0 \leq y \leq 4 \\ 0 & \text{autrement} \end{cases}
 \end{aligned}$$

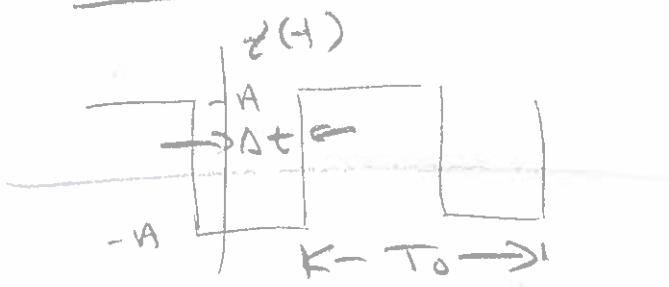
$$\begin{aligned}
 f_x(x) &= \int_0^4 f_{xy}(x, y) dy \\
 &= \int_0^4 \frac{xy}{32} dy \\
 &= \frac{xy^2}{64} \Big|_0^4 \\
 &= \frac{x(16 - x^2)}{64}
 \end{aligned}$$

$$f_x(x) = \begin{cases} \frac{16x - x^3}{64} & 0 \leq x \leq 4 \\ 0 & \text{ailleurs} \end{cases}$$

$$f_{xy}(x, y) \neq f_x(x) + f_y(y)$$

donc X et Y ne sont pas indépendantes.

Question 4



Δt est uniformément distribuée entre 0 et T_0

$$E[x(t_1)x(t_2)]$$

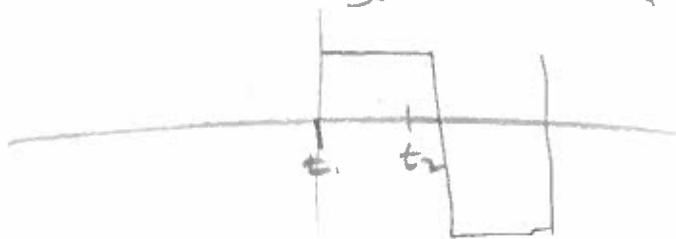
$$\text{si } t_1 = t_2 \quad E[x(t_1)x(t_2)] = A^2$$

$$\text{si } t_1 = t_2 \pm \frac{T_0}{2} \quad E[x(t_1)x(t_2)] = -A^2$$

mais si $0 \leq |t_1 - t_2| \leq \frac{T_0}{2}$ $E[x(t_1)x(t_2)]$ est trouvée par la suite

(supposons que $t_1 = 0$ et $t_2 = t_2 < \frac{T_0}{2}$)

$$\text{si } \Delta t \Rightarrow E[x(t_1)x(t_2)] = A^2$$



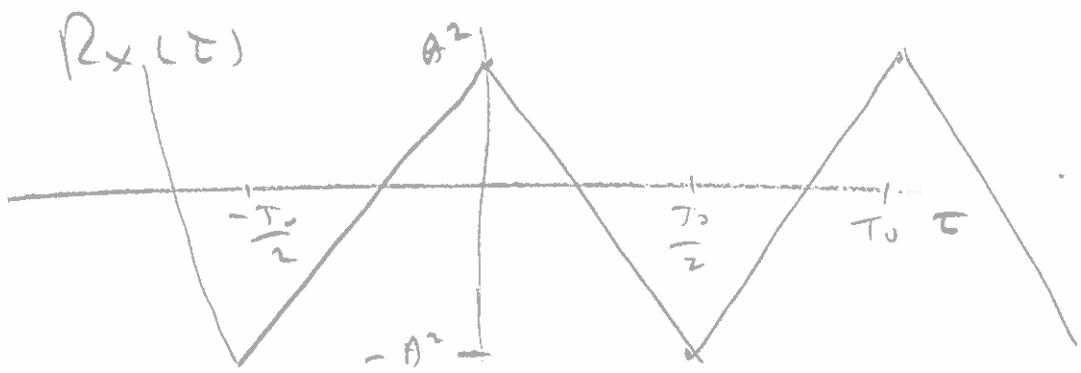
$$E[x(t_1)x(t_2)] = -A^2$$

$$\text{si } |t_1 - t_2| \rightarrow 0 \quad E[x(t_1)x(t_2)] \rightarrow A^2$$

$$\text{si } |t_1 - t_2| \rightarrow \frac{T_0}{2} \quad E[x(t_1)x(t_2)] \rightarrow -A^2$$

Quand $|t_1 - t_2|$ augmente $E[x(t_1)x(t_2)]$ est une fonction linéaire.

Parce que $x(t)$ est périodique avec période T_0 , $E[x(t)x(t-T_0)]$ est périodique avec période T_0 donc



$$R_x(\tau) = \sum_{n=-\infty}^{\infty} \frac{4A^2}{(\pi n)^2} e^{j2\pi n \tau / T_0}$$

$$S_x(f) = \sum_{n=-\infty}^{\infty} \frac{4A^2}{(\pi n)^2} \delta(f - \frac{n}{T_0})$$

question 5

$$\begin{aligned}
 R_4(\tau) &= E(X(t)X(t+\tau)) \\
 &= E(A X(t) \cos(2\pi f_0 t + \theta) A X(t+\tau) \cos(2\pi f_0 (t+\tau) + \theta)) \\
 &= A^2 E(X(t)X(t+\tau) \cos(2\pi f_0 t + \theta) \cos(2\pi f_0 (t+\tau) + \theta)) \\
 &= \frac{A^2}{2} E(X(t)X(t+\tau) \cos 2\pi f_0 \tau) \\
 &\quad + \frac{A^2}{2} E(X(t)X(t+\tau) \cos(2\pi f_0 (t+\tau) + 2\theta)) \\
 &= \frac{A^2}{2} R_x(\tau) \cos 2\pi f_0 \tau + \frac{A^2}{2} E(X(t)X(t+\tau)) \\
 &\quad \circ E(\cos 2\pi f_0 (2t+\tau) + 2\theta) \\
 &= \frac{A^2}{2} R_x(\tau) \cos 2\pi f_0 \tau
 \end{aligned}$$

↓
on a vu en
classe que
c'est 0.

ici on suppose que
 θ est indépendant
de $X(t)$