The Capacity of Gaussian MISO Channels Under Total and Per-Antenna Power Constraints

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***Abstract*—The capacity of a fixed Gaussian MIMO channel and the optimal transmission strategy under the total power (TP) constraint and full channel state information are well-known. This problem remains open in the general case under individual per-antenna (PA) power constraints, while some special cases have been solved. These include a full-rank solution for the MIMO channel and a general solution for the MISO channel. In this paper, the Gaussian MISO channel is considered and its capacity as well as optimal transmission strategies are determined in a closed form under the joint total and per-antenna power constraints in the general case. In particular, the optimal strategy is hybrid and includes two parts: first is equal-gain transmission and second is maximum-ratio transmission, which are responsible for the PA and TP constraints respectively. The optimal beamforming vector is given in a closed-form and an accurate yet simple approximation to the capacity is proposed.**

I. INTRODUCTION

The capacity of a fixed multiple-input multiple-output (MIMO) Gaussian channel under the total power (TP) constraint and full channel state information (CSI) at both ends is well-known as well as the optimal transmission strategy to achieve it [1]-[3]: the optimal strategy is Gaussian signaling over the channel eigenmodes with power allocation given by the water-filling (WF) algorithm. In the special case of multiple-input single-output (MISO) channel, this reduces to the rank-1 signalling, i.e. beamforming, where the beamforming vector is proportional to the channel vector (i.e. stronger channels get more power), which mimics the maximum ratio combining (MRC) in diversity reception systems [2], which we term here "maximum ratio transmission" (MRT). Recently, this problem was considered under individual per-antenna (PA) power constraints [4]-[6], which is motivated by the distributed design of active antenna arrays where each antenna has its own RF amplifier with limited power (as opposed to a common amplifier and a passive beamforming network in the case of TP constraint), so that powers of different antennas cannot be traded off with each other. The optimal transmission strategy for a fixed channel was established in [6], which corresponds to beamforming (i.e. rank-1 transmission) with uniform amplitude distribution across antennas and where the beamforming vector compensates for channel phase differences so that all transmitted signals are coherently combined at the receiver. This mimics the well-known equal gain combining (EGC) in a diversity-reception system. Hence, we term this strategy "equal gain transmission" (EGT) here. A fixed multiple-input multiple-output (MIMO) Gaussian channel under PA constraints was considered in [7] and [11], where a numerical algorithm to evaluate an optimal Tx covariance was developed based on a partial analytical solution [7] and a closed-form full-rank solution was obtained [11], while the general solution remains illusive. This is in stark contrast to the capacity under the TP constraint, for which the general solution is well-known for this channel. The capacity of the ergodic-fading MISO channel under the long-term average PA constraint and full CSI at both ends was established in [10].

Single-user PA-constrained results were extended to multi-user scenarios in [5] and [9], where a precoder was developed that achieves a 2-user MISO Gaussian broadcast channel (BC) capacity [5] and an iterative numerical algorithm was developed to obtain optimal covariance matrices to maximize the sum-capacity of Gaussian MIMO multiple-access (MAC) channel [9], for which no closed-form solution is known.

One may further consider a hybrid design of a Tx antenna array where each antenna has its own power amplifier and yet some power can be traded-off between antennas (corresponding to a common beamforming network) under the limited total power (e.g. due to the limitation of a power supply unit). This implies individual (PA) as well as total (TP) power constraints. Ergodic-fading MIMO channels were considered in [8] under long-term TP and short-term PA constraints and a sub-optimal signalling transmission strategy was proposed. An optimal strategy to achieve the ergodic capacity under the above constraints remains unknown. A fixed (non-fading) MISO channel was considered in [12] under full CSI at both ends and joint TP and PA constrains. It was shown that beamforming is still an optimal strategy. A closed-form solution was established in the case of 2 Tx antennas only and the general case remains an open problem.

The present paper provides a closed-form solution to this open problem, which is based on Karush-Kuhn-Tucker (KKT) optimality conditions for the respective optimization problem. In particular, we show that the optimal strategy is hybrid and consists of 2 parts: 1st part, which includes antennas with stronger channel gains and for which PA constraints are active, performs EGT (when PA constraints are the same for all antennas) while 2nd part, which includes antennas with weaker channel gains and for which PA constraints are inactive, performs MRT. This mimics the classical equal gain and maximum ratio combining (EGC and MRC) strategies of diversity reception. Amplitude distribution across antennas as well as the number of active PA constraints are explicitly determined. Sufficient and necessary conditions for the optimality of the MRT and the EGT are given. In particular, the MRT is optimal when channel gain variation among antennas is not too large and the EGT is optimal for sufficiently large total power constraint.

Based on the fact that the capacity under the joint (PA+TP) constraints is upper bounder by the capacities under the

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individual (either PA or TP) constraints, a compact yet accurate approximation to the capacity is proposed.

Notations: bold lower-case letters denote column vectors, $\boldsymbol{h} = [h_1, h_2, .., h_m]^T$, where T is transposition, while bold capital denote matrices; $\boldsymbol{R}^+$ is Hermitian conjugation of $\boldsymbol{R}$; r_{ii} denotes i-th diagonal entry of $\boldsymbol{R}$; $\lfloor x \rfloor$ is integer part while $(x)_+ = \max[0, x]$ is positive part of x; ∇_R is the derivative with respect to $\boldsymbol{R}$; $\boldsymbol{R} \geq 0$ means that $\boldsymbol{R}$ is positive semi-definite; $|\boldsymbol{h}|_p = (\sum_i |h_i|^p)^{1/p}$ is l_p-norm of vector $\boldsymbol{h}$ and $|\boldsymbol{h}| = |\boldsymbol{h}|_2$ is l_2 norm.

II. Channel Model and Capacity

Discrete-time model of a fixed Gaussian MISO channel can be put into the following form:

$$y = \boldsymbol{h}^+ \boldsymbol{x} + \xi \tag{1}$$

where $y, \boldsymbol{x}, \xi$ and $\boldsymbol{h}$ are the received and transmitted signals, noise and channel respectively; h_i^* is i-th channel gain (between i-th Tx antenna and the Rx). Without loss of generality, we order the channel gains, unless indicated otherwise, as follows: $|h_1| \geq |h_2| \geq ..|h_m| > 0$, and m is the number of transmit antennas. The noise is assumed to be Gaussian with zero mean and unit variance, so that power is also the SNR. Complex-valued channel model is assumed throughout the paper, with full channel state information available both at the transmitter and the receiver. Gaussian signaling is known to be optimal in this setting [1]-[3] so that finding the channel capacity C amounts to finding an optimal transmit covariance matrix $\boldsymbol{R}$:

$$C = \max_{\boldsymbol{R} \in S_R} \ln(1 + \boldsymbol{h}^+ \boldsymbol{R} \boldsymbol{h}) \tag{2}$$

where S_R is the constraint set. In the case of the TP constraint, it takes the form

$$S_R = \{\boldsymbol{R} : \boldsymbol{R} \geq 0, tr\boldsymbol{R} \leq P_T\}, \tag{3}$$

where P_T is the maximum total Tx power, and the MRT is optimal [2] so that the optimal covariance $\boldsymbol{R}^*$ is

$$\boldsymbol{R}^* = P_T \boldsymbol{h}\boldsymbol{h}^+ / |\boldsymbol{h}|_2^2 \tag{4}$$

and the capacity is

$$C_{MRT} = \ln(1 + P_T |\boldsymbol{h}|_2^2) \tag{5}$$

Under the PA constraints,

$$S_R = \{\boldsymbol{R} : \boldsymbol{R} \geq 0, r_{ii} \leq P\}, \tag{6}$$

where r_{ii} is i-th diagonal entry of $\boldsymbol{R}$ (the Tx power of i-th antenna), P is the maximum PA power, and the EGT is optimal [6] so that the optimal covariance $\boldsymbol{R}^*$ is

$$\boldsymbol{R}^* = P \boldsymbol{u}\boldsymbol{u}^+, \tag{7}$$

where the entries of the beamforming vector $\boldsymbol{u}$ are $u_i = e^{j\phi_i}$, ϕ_i is the phase of h_i, and the capacity is

$$C_{EGT} = \ln(1 + P |\boldsymbol{h}|_1^2) \tag{8}$$

In the next section, we consider the MISO capacity under the joint PA and TP constraints.

III. The Capacity Under the Joint Constraints

Following the same line of argument as for the total power constraint [1]-[3], the channel capacity C under the joint PA and TP constraints is as in (2) where S_R is as follows:

$$S_R = \{\boldsymbol{R} : \boldsymbol{R} \geq 0, tr\boldsymbol{R} \leq P_T, r_{ii} \leq P\} \tag{9}$$

and P_T, P are the maximum total and per-antenna powers. This is equivalent to maximizing the Rx SNR:

$$\max_{\boldsymbol{R}} \boldsymbol{h}^+ \boldsymbol{R} \boldsymbol{h} \text{ s.t. } \boldsymbol{R} \in S_R \tag{10}$$

The following Theorem gives a closed-form solution to this open problem.

Theorem 1. *The MISO channel capacity in* (2) *under the per-antenna and total power constraints in* (9) *is achieved by the beamforming with the following input covariance matrix*

$$\boldsymbol{R}^* = P^* \boldsymbol{u}\boldsymbol{u}^+ \tag{11}$$

where $P^* = \min(P_T, mP)$ *and* $\boldsymbol{u}$ *is a unitary (beamforming) vector of the form:*

$$u_i = a_i e^{j\phi_i} \tag{12}$$

where ϕ_i *is the phase of* h_i *and* a_i *represents amplitude distribution across antennas:*

$$a_i = \begin{cases} c_1, & i = 1..k \\ c_2 |h_i|, & i = k+1..m \end{cases} \tag{13}$$

and

$$c_1 = \frac{1}{\sqrt{m^*}}, \; c_2 = \frac{\sqrt{1 - k/m^*}}{|\boldsymbol{h}_{k+1}^m|_2} \tag{14}$$

$m^* = P^*/P$, $\boldsymbol{h}_{k+1}^m = [h_{k+1}...h_m]^T$ *is a truncated channel matrix, and* k *is the number of active per-antenna power constraints,* $0 \leq k \leq \lfloor m^* \rfloor$*, determined as the least solution of the following inequality*

$$|h_{k+1}| \leq h_{th} = \frac{|\boldsymbol{h}_{k+1}^m|_2}{\sqrt{m^* - k}} \tag{15}$$

if $P_T < mP$ *and* $k = m$ *otherwise. The capacity is*

$$C = \ln(1 + \gamma^*) \tag{16}$$

where $\gamma^* = \boldsymbol{h}^+ \boldsymbol{R}^* \boldsymbol{h}$ *is the maximum Rx SNR under the TP and PA constraints,*

$$\gamma^* = P^* (c_1 |\boldsymbol{h}_1^k|_1 + c_2 |\boldsymbol{h}_{k+1}^m|_2^2)^2 \tag{17}$$

where the 2nd term is absent if $k = m$.

Proof. see Appendix. □

Note from (12) that beamforming vector always compensates for channel phases so that the transmitted signals are combined coherently at the receiver, while the amplitude distribution across Tx antennas depends on the number of active PA constraints: amplitudes are always the same for those antennas for which PA constraints are active (which represent stronger channels) and they are proportional to channel gain when for inactive PA constraints (weaker channels). In accordance with this, (17) has two terms: 1st term $c_1 |\boldsymbol{h}_1^k|_1$ represents the gain due to the equal gain transmission (EGT, $|u_i| = c_1$) for active PA constraints while 2nd one $c_2|\boldsymbol{h}_{k+1}^m|_2^2$ - due to the maximum ratio transmission (MRT, $|u_i| = c_2|h_i|$) for inactive PA constraints, which mimic the equal gain combining (EGC) and maximum ratio combining (MRC) in the case of diversity reception systems.

Eq. (15) facilitates an algorithmic solution to find the number k of active PA constraints and hence the threshold h_{th}: the inequality is verified for k in increasing order, starting from $k = 0$, and the algorithm stops when 1st solution is found.

The following Corollary establishes conditions for the optimality of the MRT, which corresponds to $k = 0$.

Corollary 1. *All PA constraints are inactive and thus maximum ratio transmission is the optimal strategy if and only if*

$$|h_1| \leq |\boldsymbol{h}|_2\sqrt{P/P_T} \tag{18}$$

Proof. Follows directly from Theorem 1 by using $k = 0$. The necessary part is due to the necessity of the KKT conditions for optimality. □

Note that this limits channel gain variance among antennas. In particular, it always holds if all channel gains are the same. It also implies that at least 1 PA constraint is active if

$$|h_1| > |\boldsymbol{h}|_2\sqrt{P/P_T} \tag{19}$$

In a similar way, one obtains a condition for the optimality of the EGT.

Corollary 2. *All PA constraints are active and thus the equal gain transmission is the optimal strategy if and only if*

$$P_T \geq mP \tag{20}$$

When the TP constraint is not active, i.e. $P_T \geq mP$ and hence $k = m$, Theorem 1 reduces to the respective result in [6] under the identical PA constraints.

Examples: To illustrate the optimal solution, we consider the following example: $\boldsymbol{h} = [3, 1, 0.5, 0.1]^T$. Note that this example also applies to complex-valued channel gains since the beamforming vector is always adjusted to compensate for the channel phases (see (12)) and hence they do not affect the capacity or the amplitude distribution. Fig. 1 shows the capacity under the total and joint power constraints as the function of the total power P_T when $P = 1$. As the total power increases, more and more PA constraints become active, starting with antennas corresponding to strongest channels. Note that the MRT is optimal ($k = 0$) if the total power is not too large: $P_T \leq P|\boldsymbol{h}|^2/|h_1|^2 \approx 1.1$, while the EGT is optimal if $P_T \geq mP = 4$. Fig. 2 shows the amplitude distribution for the scenario in Fig. 1 under the joint PA+TP constraints. While weak channels get less power at the beginning (when the MRT is optimal), it gradually increases as the strongest channels reach their individual power constrains until eventually all channels have the same power (when the EGT is optimal). Note that while the amplitudes a_1 and a_4 of the strongest and weakest channels are monotonically decreasing/increasing, the amplitudes a_2, a_3 of intermediate channels are not monotonic

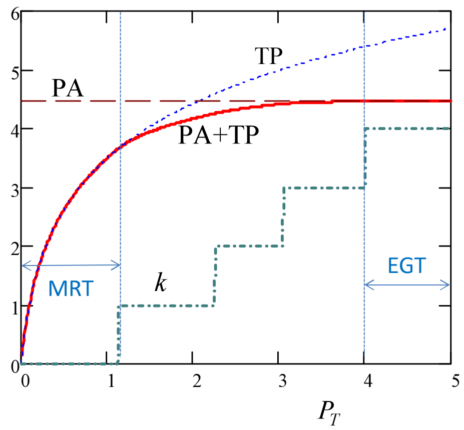

Fig. 1. The capacity of MISO channel under the PA, TP and joint PA+TP constraints and the number of active PA constraints k vs. total power P_T; $P = 1$, $\boldsymbol{h} = [3, 1, 0.5, 0.1]^T$.

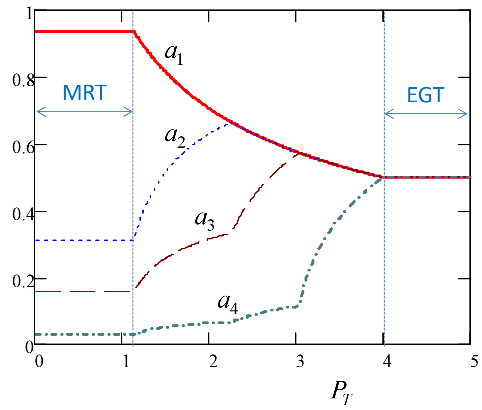

Fig. 2. The amplitude distribution under the joint power constraints for the scenario in Fig. 1.

in P_T, increasing first until they reach the stronger level and then decreasing.

In general, the capacity under the joint PA+TP constraints can be upper-bounded by the EGT and MRT capacities under the PA and TP constraints respectively:

$$C \leq \min(C_{MRT}, C_{EGT}) \tag{21}$$

where C_{MRT}, C_{EGT} are as in (5), (8), and the upper bound is tight everywhere except in the transition region, so one can approximate the capacity C as

$$C \approx \min(C_{MRT}, C_{EGT}) \tag{22}$$

It is straightforward to show that (21) and (22) hold with strict equality under (18) or (20) for any $\boldsymbol{h}$, or if $|h_1|/|h_m| = 1$ for any P_T and P. The approximation is sufficiently accurate if the variance in the channel gains is not large, i.e. if $|h_1|/|h_m|$ is not too large, as the following example demonstrates in Fig. 3.

IV. Different PA Constraints

In a similar way, one may wish to consider a more general case where individual antennas have different power con-

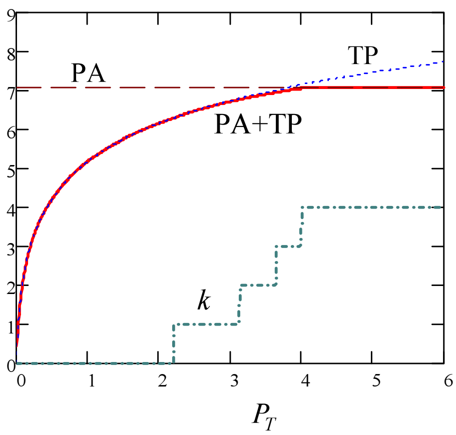

Fig. 3. The capacity of MISO channel under the PA, TP and joint PA+TP constraints and the number of active PA constraints k vs. total power P_T; $P = 1$, $\boldsymbol{h} = [4, 3, 2.5, 2]^T$. Note that the approximation in (22) is accurate over the whole range of P_T.

straints, so that the constraint set is

$$S_R = \{\boldsymbol{R} : \boldsymbol{R} \geq 0, tr\boldsymbol{R} \leq P_T, r_{ii} \leq P_i\} \tag{23}$$

The channel capacity under these constraints is given in the following.

Theorem 2. *The MISO channel capacity in* (2) *under the per-antenna and total power constraints in* (23) *is achieved by the beamforming with the input covariance matrix as in* (11) *and* (12) *where*

$$a_i = \begin{cases} c_{1i}, & i = 1..k \\ c_2|h_i|, & i = k+1..m \end{cases} \tag{24}$$

$$c_{1i} = \sqrt{\frac{P_i}{P^*}},\ c_2 = \frac{\sqrt{1 - k/m^*}}{|\boldsymbol{h}_{k+1}^m|_2} \tag{25}$$

and $P^* = \min(P_T, \sum_{i=1}^m P_i)$, $m^* = P^*/P_0$, $P_0 = \frac{1}{k}\sum_{i=1}^k P_i$ *is the average power of the active PA constraints, k is the number of active PA constraints, determined as the least solution of the following inequality*

$$\frac{|h_{k+1}|}{\sqrt{P_{k+1}}} \leq \frac{|\boldsymbol{h}_{k+1}^m|_2}{\sqrt{P_T - \sum_{i=1}^k P_i}} \tag{26}$$

if $P_T < \sum_{i=1}^m P_i$ *and* $k = m$ *otherwise, where channel gains* $\{h_i\}$ *are ordered in such a way that* $\{|h_i|/\sqrt{P_i}\}$ *are in decreasing order. The capacity is as in* (16) *and the optimal SNR is*

$$\gamma^* = P^* \left(\sum_{i=1}^k c_{1i}|h_i| + c_2|\boldsymbol{h}_{k+1}^m|_2^2 \right)^2 \tag{27}$$

Proof. Follows along the same lines as that of Theorem 1. □

Note that 1st term in (27) does not represent EGT anymore; rather, the amplitudes are adjusted to match the PA constraints. The conditions for optimality of the MRT can be similarly obtained. When the TP constraint is inactive, i.e. when $P_T \geq \sum_{i=1}^m P_i$, Theorem 2 reduces to the respective result in [6], as it should be. The condition for the optimality of the MRT is as follows.

Corollary 3. *All PA constraints are inactive and thus the MRT is optimal if and only if*

$$|h_1| \leq |\boldsymbol{h}|_2 \sqrt{P_1/P_T} \tag{28}$$

and at least 1 PA constraint is active otherwise. All PA constraints are active and hence the EGT is optimal if and only if $P_T \geq \sum_{i=1}^m P_i$.

V. Appendix

The problem in (2) under the constraints in (9) is convex (since the objective is affine and the constraints are affine and positive semi-definite). Since Slater's condition holds, KKT conditions are sufficient for optimality [13]. The Lagrangian for this problem is:

$$L = -\boldsymbol{h}^+\boldsymbol{R}\boldsymbol{h} + \lambda(tr\boldsymbol{R} - P_T) + \sum_i \lambda_i(r_{ii} - P) - tr\boldsymbol{M}\boldsymbol{R}$$

where $\lambda, \lambda_i \geq 0$ are Lagrange multipliers responsible for the total and per-antenna power constraints, and $\boldsymbol{M} \geq 0$ is (matrix) Lagrange multiplier responsible for the positive semi-definite constraint $\boldsymbol{R} \geq 0$. The KKT conditions are

$$\nabla_R L = -\boldsymbol{h}\boldsymbol{h}^+ + \lambda\boldsymbol{I} - \boldsymbol{M} + \boldsymbol{\Lambda} = 0 \tag{29}$$

$$\lambda(tr\boldsymbol{R} - P_T) = 0,\ \lambda_i(r_{ii} - P) = 0,\ \boldsymbol{R}\boldsymbol{M} = 0 \tag{30}$$

$$tr\boldsymbol{R} \leq P_T,\ r_{ii} \leq P, \tag{31}$$

$$\boldsymbol{M} \geq 0,\ \lambda_i \geq 0 \tag{32}$$

where ∇_R is the derivative with respect to $\boldsymbol{R}$ and $\boldsymbol{\Lambda} = diag\{\lambda_1 ... \lambda_m\}$ is a diagonal matrix collecting λ_i; (29) is the stationarity condition, (30) are complementary slackness conditions; (31) and (32) are primal and dual feasibility conditions.

Combining both inequalities in (31), one obtains:

$$tr\boldsymbol{R} \leq \min(P_T, mP) = P^* \tag{33}$$

and from (29)

$$\boldsymbol{h}\boldsymbol{h}^+ + \boldsymbol{M} = \lambda\boldsymbol{I} + \boldsymbol{\Lambda} > 0 \tag{34}$$

where the last inequality is due to the diagonal part of the equality:

$$|h_i|^2 + m_{ii} = \lambda + \lambda_i > 0 \tag{35}$$

since $m_{ii} \geq 0$ and $|h_i| > 0$. Therefore, $\boldsymbol{h}\boldsymbol{h}^+ + \boldsymbol{M}$ is full-rank, $r(\boldsymbol{h}\boldsymbol{h}^+ + \boldsymbol{M}) = m$. Since $r(\boldsymbol{h}\boldsymbol{h}^+) = 1$ and $\boldsymbol{M} \geq 0$, it follows that $r(\boldsymbol{M}) \geq m - 1$. Since $r(\boldsymbol{M}) = m$ implies $\boldsymbol{M} > 0$ and hence $\boldsymbol{R} = 0$ - clearly not an optimal solution, one concludes that $r(\boldsymbol{M}) = m - 1$ and hence $r(\boldsymbol{R}) = 1$ (this follows from complementary slackness $\boldsymbol{M}\boldsymbol{R} = 0$), i.e. beamforming is optimal:

$$\boldsymbol{R}^* = P^*\boldsymbol{u}\boldsymbol{u}^+ \tag{36}$$

where $|\boldsymbol{u}| = 1$. It remains to find the beamforming vector $\boldsymbol{u}$. To this end, combining the last equation with $\boldsymbol{M}\boldsymbol{R} = 0$, one obtains:

$$0 = \boldsymbol{M}\boldsymbol{u} = -\boldsymbol{h}^+\boldsymbol{u}\boldsymbol{h} + (\boldsymbol{\Lambda} + \lambda\boldsymbol{I})\boldsymbol{u} \tag{37}$$

from which it follows that

$$u_i = \boldsymbol{h}^+ \boldsymbol{u} h_i/(\lambda + \lambda_i) \tag{38}$$

and hence

$$\phi_{ui} = \phi_i + \varphi = \phi_i \tag{39}$$

where ϕ_{ui}, φ are the phases of u_i and $\boldsymbol{h}^+\boldsymbol{u}$; since the common phase φ does not affect $\boldsymbol{R}$ or the SNR, one can set $\varphi = 0$ without loss of generality to obtain

$$u_i = a h_i/(\lambda + \lambda_i) \tag{40}$$

where $a = |\boldsymbol{h}^+\boldsymbol{u}|$.

If $\lambda_i > 0$ (active i-th per-antenna constraint), then $r_{ii} = P^*|u_i|^2 = P$ from (30) and (11) so that $|u_i| = c_1 = 1/\sqrt{m^*}$. Since $\lambda_i > 0$, using (40),

$$c_1 = |u_i| = a|h_i|/(\lambda + \lambda_i) < a|h_i|/\lambda \tag{41}$$

so that

$$|h_i| > \lambda c_1/a = h_{th} \tag{42}$$

where h_{th} is a threshold channel gain, i.e. PA constraints are active for all sufficiently strong channels.

When $\lambda_i = 0$ (inactive i-th PA constraint) for at least one i, it follows from (35) that $\lambda > 0$, i.e. the TP constraint is active: $tr\boldsymbol{R} = P_T$, which implies $P_T \leq mP$. One obtains from (40) in this case

$$u_i = c_2 h_i, \ c_2 = a/\lambda \tag{43}$$

which, when combined with the PA constraint $r_{ii} = P_T|u_i|^2 \leq P$, requires

$$|h_i| \leq h_{th} \tag{44}$$

where c_2 can be found from the TP constraint $|\boldsymbol{u}|^2 = 1$:

$$|\boldsymbol{u}|^2 = kc_1^2 + c_2^2|\boldsymbol{h}_{k+1}^m|^2 = 1 \tag{45}$$

and $k < m$ is the number of active PA constraints, i.e. when (42) holds, which implies

$$c_2 = \frac{\sqrt{1 - k/m^*}}{|\boldsymbol{h}_{k+1}^m|_2} \tag{46}$$

so that $k \leq m^*$ and h_{th} can be expressed as

$$h_{th} = \frac{\lambda c_1}{a} = \frac{c_1}{c_2} = \frac{|\boldsymbol{h}_{k+1}^m|_2}{\sqrt{m^* - k}} \tag{47}$$

If $k = m$, i.e. all PA constraints are active, then one can take $h_{th} = 0$ for consistency with (42). This implies $P_T \geq mP$ so that $m^* = m$ (note that (47) is not defined in this case).

To find the number k of active PA constraints when $P_T < mP$, so that $m^* = P_T/P < m$ and hence $k \leq m^* < m$, observe that (42) and (43) imply

$$|h_k|\sqrt{m^* - k} > |\boldsymbol{h}_{k+1}^m|_2 \tag{48}$$

while (44) implies

$$|h_{k+1}|\sqrt{m^* - k} \leq |\boldsymbol{h}_{k+1}^m|_2 \tag{49}$$

both due to the ordering $|h_1| \geq |h_2| \geq .. \geq |h_m|$, so that k has to satisfy both inequalities simultaneously. The following Proposition shows that a unique k satisfying both of them does exist.

Proposition 1. *There exists a unique solution of* (48) *and* (49)*, which is also the least solution of* (49).

Proof. We need the following technical Lemmas.

Lemma 1. *There exists at least one solution k of* (49).

Proof. If $m^* = m$, then $k = m$ clearly solves it, where we take $h_{m+1} = 0$ for consistency (recall that all channels with 0 gain do not affect the capacity). If $m^* < m$, then $k = \lfloor m^* \rfloor$ solves it. □

Lemma 2. *If $k \leq \lfloor m^* \rfloor$ satisfies* (49)*, then all k' such that $k \leq k' \leq \lfloor m^* \rfloor$ also satisfy it, i.e. a solution is not unique in general. Likewise, all $k' \leq k$ solve* (48) *if k solves it.*

Now note, from Lemma 1, that a least solution k' of (49) exists, so that the following holds

$$|h_{k'+1}|^2(m^* - k') \leq |h_{k'+1}|^2 + .. + |h_m|^2 \tag{50}$$

$$|h_{k'}|^2(m^* - k + 1) > |h_{k'}|^2 + .. + |h_m|^2 \tag{51}$$

where the last inequality is due to the fact that k' is the least solution; this inequality implies

$$|h_{k'}|^2(m^* - k) > |h_{k'+1}|^2 + .. + |h_m|^2 \tag{52}$$

i.e. (48) holds for $k = k'$. □

Using (29) in combination with (39)-(47), it can be further shown that $\boldsymbol{M} \geq 0$, i.e. the dual feasibility holds. This completes the proof.

References

[1] T.M. Cover, J.A. Thomas, Elements of Information Theory, Wiley, 2006.

[2] D. Tse and P. Viswanath, Fundamentals of Wireless Communication, Cambridge University Press, 2005.

[3] E. Biglieri, J. Proakis, and S. Shamai, "Fading Channels: Information-Theoretic and Communications Aspects," IEEE Trans. Inform. Theory, vol. 44, No. 6, pp. 2619-2692, Oct. 1998.

[4] W. Yu and T. Lan, Transmitter optimization for the multi-antenna downlink with per-antenna power constraint, IEEE Trans. Signal Process., vol.55, no. 6, pp. 2646-2660, June 2007.

[5] J. Park, W. Sung, and T. Duman, Precoder and capacity expressions for optimal two-user MIMO transmission with per-antenna power constraints, IEEE Comm. Letters, vol. 14, no. 11, pp. 996-998, Nov. 2010

[6] M. Vu, MISO Capacity with Per-antenna power constraint, IEEE Trans. on Commun., vol. 59, no. 5, May 2011.

[7] M. Vu, MIMO Capacity with Per-Antenna Power Constraint, IEEE Globecom, Houston, USA, 5-9 Dec., 2011.

[8] M. Khoshnevisan, J.N. Laneman, Power Allocation in Multi-Antenna Wireless Systems Subject to Simultaneous Power Constraints, IEEE Trans. Comm., v.60, No. 12, pp. 3855–3864, Dec. 2012.

[9] Y. Zhu, M. Vu, Iterative Mode-Dropping for the Sum Capacity of MIMO-MAC with Per-Antenna Power Constraint, IEEE Trans. Comm., v. 60, N. 9, pp. 2421–2426, Sep. 2012.

[10] D. Maamari, N. Devroye, D. Tuninetti, The Capacity of the Ergodic MISO Channel with Per-antenna Power Constraint and an Application to the Fading Cognitive Interference Channel, IEEE Int. Symp. Information Theory, Hawaii, USA, July 2014.

[11] D. Tuninetti, "On the capacity of the AWGN MIMO channel under per-antenna power constraints", 2014 IEEE International Conference on Communications (ICC), Sydney, June 2014.

[12] P. Cao et al, "Optimal Transmission Rate for MISO Channels with Joint Sum and Per-antenna Power Constraints", IEEE International Conference on Communications (ICC), London, June 08-12, 2015.

[13] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.