On The Outage Capacity Distribution of Correlated Keyhole MIMO Channels

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Abstract – **Keyhole MIMO channels, which were predicted theoretically and also observed experimentally, have recently received significant attention as they may appear in some practically-important propagation scenarios. This paper concentrates on a capacity study of such channels. Closed-form expressions for the instantaneous SNR and outage capacity distributions of a spatially correlated keyhole MIMO channel are given. The case of non-singular correlation matrices with distinct eigenvalues is considered in detail. When the number of Tx (Rx) antennas is large, the correlated keyhole channel tends asymptotically to the Rayleigh diversity channel with a single Tx (Rx) and multiple Rx (Tx) antennas. The outage capacity at low outage probabilities and the diversity order of the keyhole channel is upper-bounded by that of the equivalent Rayleigh diversity channel. The asymptotic outage capacity distribution, when the numbers of Tx and Rx antennas are both large, is Gaussian under general conditions on the correlation (the average SNR affects the mean and the correlation affects the variance). The Gaussian approximation is accurate already for a reasonably small number of antennas. Using the single-parameter exponential correlation matrices, we show that the outage capacity at low outage probabilities decreases with correlation.**

Index Terms - MIMO systems, keyhole channel, outage capacity distributio, correlation.

I. Introduction

Multiple-Input-Multiple-Output (MIMO) systems have become an attractive solution in wireless communications due to enormously large spectral efficiency. One of the major statistical characteristics of the MIMO channel in multipath environment is its outage capacity, which gives the ultimate upper limit on the error-free information rate with a given probability of outage [1]. The outage capacity distribution of different channels has been extensively studied, and many analytical and empirical results have been obtained. For example, a non-correlated and correlated Rayleigh MIMO channels are well studied and closed-form expressions for their outage capacity distributions have been found [2]. Many measurement-based works show that the wide range of real channels follows closely those analytical results [3].

On the contrary, the outage capacity distribution of keyhole MIMO channels has not been studied in sufficient depth. The keyhole channel was analytically predicted in [4]. It can be modeled as a concatenation of two Rayleigh sub-channels separated by a keyhole whose dimensions are much smaller than the wavelength. As shown in [4], the presence of the keyhole degenerates the channel, i.e. its rank is one regardless the number of Tx and Rx antennas. Consequently, the capacity of such channels deteriorates significantly comparing to the Rayeigh channels with the same number of Tx and Rx antennas. There is a significant interest in the keyhole channels in recent literature as they may appear in some practically important propagation scenarios. Chizhik *et al* [5] suggests a keyhole scenario where the link between Tx and Rx ends is due to the 1-D diffraction. There is a number of experimental evidences of a keyhole channel. The measurements of the channel capacity along a hallway, reported in [6], show the decrease in capacity with distance, which is explained by the keyhole effect in the hallways. Almers *et al* [7] show that the keyhole model describes well wireless channels when the wave propagates via narrow tunnels such as waveguides. However, the literature dealing with the information theoretic analysis of such channels is rather limited. Closed-form expressions for the mean (ergodic) capacity of a spatially uncorrelated keyhole channel are presented in [8]. Performance analysis of space-time block codes over an uncorrelated keyhole channel is shown in [9], where, in particular, the moment generating function of the instantaneous after-decoding signal to noise ratio (SNR) is derived, and the symbol error rates (SER) for various codes are evaluated. An asymptotically tight lower bound on the mean capacity of a spatially correlated keyhole channel is proposed in [10].

These papers, however, do not consider the outage capacity, which is a more relevant performance measure in a fading channel from a practical perspective (i.e. for a given quality of service) as compared to the mean capacity. To fill the gap, the present paper derives closed-form expressions for the instantaneous SNR and the outage capacity distributions of correlated keyhole MIMO channels. We consider a particular but the most common case where the correlation matrices at the Tx and Rx ends are non-singular and have distinct eigenvalues. We show that the keyhole channel is statistically different from traditional diversity channels, which also have rank one. However, when the number of either Tx or Rx antennas is large, the keyhole channel tends asymptotically to the Rayleigh diversity channel with a single Tx or Rx antenna respectively. The capacity distribution at low outage probabilities and the

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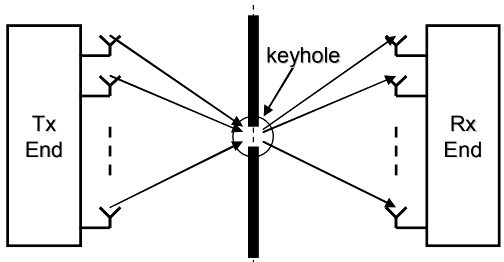

Fig. 1. A keyhole MIMO channel. Each end has rich multipath so that the sub-channels are correlated Rayleigh fading.

diversity order of the keyhole channel are upper-bounded by those of the equivalent Rayleigh diversity channel.

Since the expression for the exact capacity distribution is rather complicated and does not allow for significant insight, we use the asymptotic analysis, which has already been successfully exploited in [11] and [12] for the Rayleigh MIMO channels, to derive the asymptotic outage capacity distribution of keyhole MIMO channels when the number of Tx and Rx antennas is large. We show that, under certain general conditions on correlation, this distribution is Gaussian; the average SNR affects the mean, which is independent of the correlation, and the correlation affects the variance. In order to study the impact of the correlation on the outage capacity in explicit form, we consider the exponential and quadratic exponential correlation matrix models [13, 14] and show analytically that the larger the correlation the larger the variance of the asymptotic capacity. In turn, the larger variance results in smaller capacity at low outage probabilities.

Finally, we demonstrate that the exact capacity distribution follows closely the asymptotic one already for a reasonably small number of Tx and Rx antennas; the discrepancy is insignificant from the practical point of view. Hence, the simple asymptotical expression not only offers a significant insight, but also can be applied to practical problems.

II. KEYHOLE MIMO CHANNEL CAPACITY

Consider a spatially correlated keyhole MIMO channel with n_t Tx and n_r Rx antennas (see Fig. 1). Let the element H_{km}, $k=1..n_t; m=1...n_r$, of the channel transfer matrix $\mathbf{H}$ be a complex channel gain from the m^{th} transmit to the k^{th} receive antenna. The gain matrix of a keyhole channel is given by [4]:

$$\mathbf{H} = \mathbf{h}_r \mathbf{h}_t^H \tag{1}$$

where $(\ )^H$ denotes the Hermitian transpose, $\mathbf{h}_t\ [n_t \times 1]$ and $\mathbf{h}_r\ [n_r \times 1]$ are mutually independent random vectors representing the complex gains from the transmit antennas to the keyhole and from the keyhole to the receive antennas respectively. Since the considered keyhole channel is a concatenation of the two correlated Rayleigh sub-channels, $\mathbf{h}_t$ and $\mathbf{h}_r$ are complex circular symmetric correlated Gaussian vectors with zero means and correlation matrices $\mathbf{R}_t = E\{\mathbf{h}_t \mathbf{h}_t^H\}$ and $\mathbf{R}_r = E\{\mathbf{h}_r \mathbf{h}_r^H\}$ respectively, where $E\{\}$ denotes expectation. $\mathbf{H}$ is normalized such that $E\{\|\mathbf{H}\|^2\} = n_t n_r$, where $\|\ \|$ is the L_2 norm, and $n_t^{-1} E\{\|\mathbf{h}_t\|^2\} = n_r^{-1} E\{\|\mathbf{h}_r\|^2\} = 1$, which also implies $n_t^{-1} trace\{\mathbf{R}_t\} = n_r^{-1} trace\{\mathbf{R}_r\} = 1$.

From [1], when the channel state information (CSI) is available at the Rx but not the Tx end, the instantaneous capacity (i.e. the capacity for a given channel realization) of a quasi-static frequency flat MIMO channel in natural units $[nat]$ is given by:

$$C = \ln\left(\det[\mathbf{I} + \gamma_0 \mathbf{H}\mathbf{H}^H / n_t]\right) \tag{2}$$

where $\det$ is the determinant, $\mathbf{I}$ is $[n_r \times n_r]$ identity matrix and γ_0 is the average SNR per Rx antenna. Substituting (1) in (2) and using the fact that for any suitable matrices $\mathbf{A}$ and $\mathbf{B}$, $\det[\mathbf{I} + \mathbf{AB}] = \det[\mathbf{I} + \mathbf{BA}]$, it is straightforward to show that the instantaneous capacity of the keyhole channel is:

$$C = \ln\left(1 + \frac{\gamma_0}{n_t} \|\mathbf{h}_t\|^2 \|\mathbf{h}_r\|^2\right) = \ln\left(1 + \frac{\gamma_0}{n_t}\alpha\right) \tag{3}$$

Following (3), a technique to achieve the capacity is to implement maximum ratio combining (MRC) at the Tx and Rx ends such that the different signals from the Tx antennas would add up coherently at the keyhole and also at the Rx end. Based on this concept $\alpha = \|\mathbf{h}_t\|^2 \|\mathbf{h}_r\|^2$ in (3) is a normalized instantaneous SNR of the combined signal at the receiver. Since the instantaneous capacity is a continuous, monotonically increasing function of α, the cumulative distribution function (CDF) of C, which is also the outage capacity distribution $F_C(x)$, is given by:

$$F_C(x) = F_\alpha\left(n_t (e^x - 1)/\gamma_0\right) \tag{4}$$

where $F_\alpha(x)$ is the CDF of α. The exact expressions for $F_\alpha(x)$ and consequently for $F_C(x)$ are derived in the next section.

III. EXACT OUTAGE CAPACITY DISTRIBUTION

Since α is a product of two mutually independent non-negative random variables $\beta_t = \|\mathbf{h}_t\|^2$ and $\beta_r = \|\mathbf{h}_r\|^2$, the probability density function (PDF) $f_\alpha(z)$, CDF $F_\alpha(z)$ and characteristic function (CF) $\Phi_\alpha(\omega)$ of α are given by:

$$f_\alpha(z) = \int_0^\infty f_{\beta_t}(z/x) f_{\beta_r}(x)\, d\ln(x) \tag{5}$$

$$F_\alpha(z) = \int_0^\infty F_{\beta_t}(z/x) f_{\beta_r}(x)\, dx \tag{6}$$

$$\Phi_\alpha(\omega) = \int_0^\infty \Phi_{\beta_t}(x\omega) f_{\beta_r}(x)\, dx \tag{7}$$

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where $F_{\beta_t}(x), f_{\beta_t}(x), \Phi_{\beta_t}(\omega)$ and $F_{\beta_r}(x)$, $f_{\beta_r}(x)$, $\Phi_{\beta_r}(\omega)$ are the CDF, PDF and CF of β_t and β_r respectively.

Let β be either β_t or β_r. β has the generalized χ^2 distribution with characteristic function $\Phi_\beta(\omega) = \det^{-1}[\mathbf{I} - j\omega\mathbf{R}]$ where $\mathbf{R}$ is either $\mathbf{R}_t$ or $\mathbf{R}_r$ and $j = \sqrt{-1}$. Consider a particular case where $\mathbf{R}$ is non-singular and has n distinct eigenvalues λ_k, the CF of β can be represented as:

$$\Phi_\beta(\omega) = \prod_{k=1}^{n}(1 - j\omega\lambda_k)^{-1} = \sum_{k=1}^{n} A_k (1 - j\omega\lambda_k)^{-1} \quad (8)$$

where A_k are the coefficients of the partial-fraction-decomposition of $\Phi_\beta(\omega)$. It can be shown that when λ_k are distinct

$$A_k = \prod_{\substack{m=1 \\ m \neq k}}^{n} (1 - \lambda_m / \lambda_k)^{-1} \quad (9)$$

The derivation of (9) is rather simple and it is omitted due to the page limit. From (8) the CDF of β is given by:

$$F_\beta(x) = \sum_{k=1}^{n} A_k (1 - \exp\{-x/\lambda_k\}) \quad (10)$$

Using (5), (6) and (10), the PDF and CDF of α are obtained as:

$$f_\alpha(x) = 2\sum_{k=1}^{n_t}\sum_{m=1}^{n_r} \frac{A_k^t A_m^r}{\lambda_k^t \lambda_m^r} K_0\left(\sqrt{\frac{4x}{\lambda_k^t \lambda_m^r}}\right) \quad (11)$$

$$F_\alpha(x) = 1 - \sum_{k=1}^{n_t}\sum_{m=1}^{n_r} A_k^t A_m^r \sqrt{\frac{4x}{\lambda_k^t \lambda_m^r}} K_1\left(\sqrt{\frac{4x}{\lambda_k^t \lambda_m^r}}\right) \quad (12)$$

where A_k^t and A_m^r are the coefficients of the partial-fraction-decomposition of $\Phi_{\beta_t}(\omega)$ and $\Phi_{\beta_r}(\omega)$ respectively, λ_k^t and λ_m^r are the eigenvalues of $\mathbf{R}_t$ and $\mathbf{R}_r$, and $K_n(x)$ is the n-order modified Bessel function of the second kind. Finally, by substituting (12) in (4) the exact expression for the outage capacity distribution of the spatially correlated keyhole MIMO channel is obtained. Eq. (11), (12) and (4) have been validated by extensive Monte-Carlo (MC) simulations for various n_t, n_r, γ_0, $\mathbf{R}_t$ and $\mathbf{R}_r$. As an example, Fig. 2 shows the analytical outage capacity distribution (4) and MC simulated one using 10,000 trials for $\gamma_0 = 20dB$ and $n_t \times n_r =$ 2x2, 4x4 and 8x8. $\mathbf{R}_t$ and $\mathbf{R}_r$ were modeled using exponential correlation matrix [13] with correlation parameters $r_t = 0.5$ and $r_r = 0.8$ at Tx and Rx ends respectively. In all considered cases the agreement between the simulation and analytical distribution is good. The maximum deviation of the simulated distribution is within the $\pm\sigma$ error range, where σ is a standard deviation of the simulated distribution due to the finite statistics (10,000 trials) (see [3] for details on statistical

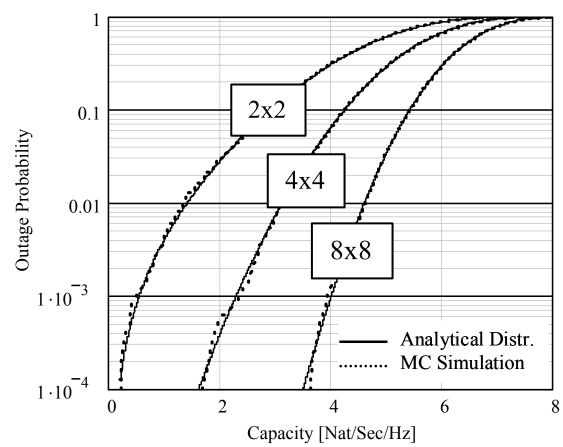

Fig. 2. Outage capacity of a keyhole channel: analytical distribution vs. MC simulation.

analysis). The $\pm\sigma$ boundaries are not shown for the figure clarity.

It is interesting to consider asymptotic outage capacity distribution of the keyhole MIMO channels. For this we need the following lemma. A proof is omitted due to the page limit.

Lemma 1: Let β be a generalized χ^2 random variable with CF $\Phi_\beta(\omega) = \det^{-1}[\mathbf{I} - j\omega\mathbf{R}]$, where $\mathbf{R}$ is an $n \times n$ correlation matrix. If $\lim_{n\to\infty} n^{-1} trace\{\mathbf{R}\} < \infty$ and $\lim_{n\to\infty} n^{-2}\|\mathbf{R}\|^2 = 0$, the distribution of $n^{-1}\beta$ is asymptotically Gaussian as $n \to \infty$ with mean $\mu = n^{-1} trace\{\mathbf{R}\}$ and variance $\sigma^2 = n^{-2}\|\mathbf{R}\|^2$.

Using (7), it can be shown that when either n_t or n_r tends to infinity and either $\mathbf{R}_t$ or $\mathbf{R}_r$ satisfies the conditions of Lemma 1, the asymptotic CF's of α are:

$$\Phi_\alpha(\omega) \to \det^{-1}(\mathbf{I} - j\omega\mathbf{R}_r n_t), \text{ as } n_t \to \infty \quad (13)$$

$$\Phi_\alpha(\omega) \to \det^{-1}(\mathbf{I} - j\omega\mathbf{R}_t n_r), \text{ as } n_r \to \infty \quad (14)$$

A proof follows directly form Lemma 1 and it is omitted due to the page limit. Eq. (13) and (14) mean that the asymptotic distributions of α are identical to those of $n_t\beta_r$ and $n_r\beta_t$ respectively. Therefore following (3), the asymptotic instantaneous capacities of the keyhole channel are given by:

$$C \xrightarrow{d} \ln(1 + \gamma_0\beta_r), \text{ as } n_t \to \infty \quad (15)$$

$$C \xrightarrow{d} \ln\left(1 + \frac{\gamma_0 n_r}{n_t}\beta_t\right) \text{ as } n_r \to \infty \quad (16)$$

where $\xrightarrow{d}$ means convergence in distribution. Note that the right side of (15) is explicitly the instantaneous capacity of the $1 \times n_r$ Rayleigh channel and (16) is that of the $n_t \times 1$ Rayleigh

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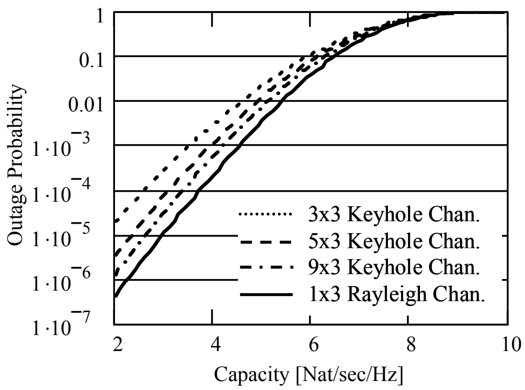

Fig. 3. Outage capacity distributions: *n*x3 keyhole channels vs. 1x3 Rayleigh channel.

channel with the Rx antenna gain equal to n_r. This asymptotic behavior of the keyhole channel is physically explained by the fact that when $n_t \to \infty$ the Raleigh sub-channel at the Tx end is asymptotically Gaussian with one equivalent Tx antenna, which corresponds to the Rx diversity channel. Similarly, when $n_r \to \infty$ the sub-channel at the Rx end is asymptotically Gaussian with one Rx antenna, which corresponds to the Tx diversity channel. This fact has been proven in [9] for uncorrelated keyhole channels, eq. (15) and (16) show that the same is true for the correlated channels under the conditions of Lemma 1. Since (15) and (16) are both continuous, monotonically increasing functions of β_r and β_t respectively and the CDF's of $F_{\beta_t}(x)$ and $F_{\beta_r}(x)$ are given by (10), the asymptotic outage capacity distributions of the keyhole channel for $n_t \to \infty$ and $n_r \to \infty$ are given respectively by:

$$F_C(x) \to F_{\beta_r}\left(\gamma_0^{-1}(e^x - 1)\right), \text{ as } n_t \to \infty \tag{17}$$

$$F_C(x) \to F_{\beta_t}\left(\gamma_0^{-1}(e^x - 1)\frac{n_t}{n_r}\right), \text{ as } n_r \to \infty \tag{18}$$

Since the increase in the number of Tx or Rx antennas can not decrease the mean capacity or diversity order of a MIMO channel, and since the keyhole channel tends asymptotically to the $1 \times n_r$ or $n_t \times 1$ Rayleigh diversity channels when either n_t or n_r increases, the capacity at low outage probabilities and diversity order of the equivalent Rayleigh diversity channels upper-bound those of the keyhole channel. To illuminate this fact, the outage capacity distributions of different $n_t \times 3$ keyhole and 1x3 equivalent Rayleigh channels are plotted in Fig. 3 for $\gamma_0 = 30dB$. $\mathbf{R}_t$ and $\mathbf{R}_r$ were modeled using the exponential correlation matrix with correlation parameters $r = 0.7$ at both Tx and Rx ends. Clearly, as the number of Tx antennas increases, the outage capacity of the corresponding keyhole channels approaches asymptotically that of the equivalent 1x3 Rayleigh channel.

Eq. (4) for the exact outage capacity distribution has a complicated form, which makes it difficult to get insight and to evaluate the effect of various parameters on the capacity. In particular the impact of correlation on the capacity distribution is difficult to see. Even though the asymptotic distributions above underlines the relationship between the keyhole channel and the equivalent Rayleigh diversity channels, eq. (17) and (18) are still intricate and do not contribute much to that understanding. Moreover, when n_t or n_r are large and the correlation at either Tx or Rx end is low, the partial fraction decomposition coefficients (9) are very large and the numerical calculation of (4) suffers from the loss of precision. To overcome all these problems, we derive below the asymptotic outage capacity distribution of the correlated keyhole channel when both n_t and n_r are large.

IV. ASYMPTOTIC OUTAGE CAPACITY DISTRIBUTION

We begin with the following theorem, which is important for further discussion.

Theorem 1: Let C be an instantaneous capacity of the correlated keyhole channel given in (3); $\mathbf{R}_t = E\{\mathbf{h}_t\mathbf{h}_t^H\}$ and $\mathbf{R}_r = E\{\mathbf{h}_r\mathbf{h}_r^H\}$. When both n_t and n_r tend to infinity, the distribution of C is Gaussian in probability if $\lim_{n_t\to\infty} n_t^{-1} trace\{\mathbf{R}_t\}$ and $\lim_{n_r\to\infty} n_r^{-1} trace\{\mathbf{R}_r\}$ are finite and $\lim_{n_t\to\infty} n_t^{-2}\|\mathbf{R}_t\|^2 = \lim_{n_r\to\infty} n_r^{-2}\|\mathbf{R}_r\|^2 = 0$. Moreover, if the channel is normalized so that $\lim_{n_t\to\infty} n_t^{-1} trace\{\mathbf{R}_t\} = 1$ and $\lim_{n_r\to\infty} n_r^{-1} trace\{\mathbf{R}_r\} = 1$, the asymptotic mean μ and the variance σ^2 of C are as follows:

$$\mu = \ln(1 + n_r\gamma_0); \sigma^2 = n_t^{-2}\|\mathbf{R}_t\|^2 + n_r^{-2}\|\mathbf{R}_r\|^2 \tag{19}$$

A proof is based on Lemma 1 and it is omitted due to the page limit. The conditions of Theorem 2 do not require either distinct eigenvalues of the correlation matrices $\mathbf{R}_t$ and $\mathbf{R}_r$, or $\det[\mathbf{R}_t] \neq 0$, $\det[\mathbf{R}_r] \neq 0$. Therefore the outage capacity distribution of an uncorrelated keyhole MIMO channel is asymptotically Gaussian as well, with the mean as in (19) and variance $\sigma^2 = n_t^{-1} + n_r^{-1}$. As follows from (19), it is not necessarily true that increase in the number of antennas decreases the variance and hence, the outage probability, but only if σ^2 is monotonically decreasing with n_t and n_r, i.e. $\|\mathbf{R}_t\|$ and $\|\mathbf{R}_r\|$ increase not faster than $n_t^{1-\varepsilon_1}$ and $n_r^{1-\varepsilon_2}$ respectively for some $\varepsilon_1, \varepsilon_2 > 0$.

Since μ is a function of n_r and γ_0 only, the immediate conclusion is that correlation has no effect on the asymptotic mean capacity, but only on the variance. In contrast, increase in the average SNR increases the mean capacity while the variance remains unchanged. We note that the asymptotic mean capacity in (19) and the upper bound on the mean capacity of the finite order keyhole channel proposed in [8] are identical.

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Unlike the exact distribution, the asymptotic one is expressed through the simple functions of matrices $\mathbf{R}_t$ and $\mathbf{R}_r$ such as trace and norm rather than through eigenvalues and coefficients of the partial-fraction-decomposition. This makes the analysis simpler. Further we consider two single-parameter correlation matrix models for $\mathbf{R}_t$ and $\mathbf{R}_r$ to show explicitly the impact of correlation on the asymptotic capacity distribution.

First, we consider the exponential correlation model [13], where the elements of correlation matrix $\mathbf{R}$, either $\mathbf{R}_t$ or $\mathbf{R}_r$, are represented through a single complex correlation parameter r as following:

$$R_{km} = \begin{cases} r^{m-k}; & m \geq k \\ \bar{r}^{k-m}; & m < k \end{cases}, \quad |r| < 1 \tag{20}$$

where $\bar{r}$ is the complex conjugate of r. This model allows for significant insight and has been successfully used for many communications problems. Despite its simplicity, it is a physically-reasonable model in the sense that the correlation decreases as the distance between antennas increases. It can be shown that $\mathbf{R}$ in (20) satisfies the conditions of Lemma 1 and consequently those of Theorem 2,

$$\lim_{n\to\infty} n^{-1} trace\{\mathbf{R}\} = 1; \ \lim_{n\to\infty} n^{-2} \|\mathbf{R}\|^2 = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1+|r|^2}{1-|r|^2} = 0 \tag{21}$$

A proof is based on the convergence properties of the geometric series and is omitted due to the page limit. Therefore, when both $\mathbf{R}_t$ and $\mathbf{R}_r$ are given by the exponential model the asymptotic capacity distribution of such a keyhole channel is Gaussian with the mean as in (19) and the variance given by

$$\sigma^2 = \frac{1}{n_t} \cdot \frac{1+|r_t|^2}{1-|r_t|^2} + \frac{1}{n_r} \cdot \frac{1+|r_r|^2}{1-|r_r|^2} \tag{22}$$

where r_t and r_r are the correlation parameters in $\mathbf{R}_t$ and $\mathbf{R}_r$ respectively. To get some insight, assume that $n = n_t = n_r$ and $r = r_t = r_r$, then the asymptotic capacity distributions of an uncorrelated $r = 0$ and correlated $r \neq 0$ keyhole channels have the same mean $\ln(1 + n\gamma_0)$ but different variances σ_u^2 and σ_c^2:

$$\eta' \overset{\Delta}{=} \frac{\sigma_c^2}{\sigma_u^2} = \frac{1+|r|^2}{1-|r|^2} \tag{23}$$

For any $|r| < 1$, $\sigma_c^2 \geq \sigma_u^2$ i.e. $\eta' \geq 1$. Moreover, η' is a monotonically increasing function of $|r|$. Therefore, the outage capacity of the uncorrelated channel is larger than that of the correlated one for the same outage probability $P_{out} < 0.5$. The larger $|r|$, the larger the gap between the two capacities.

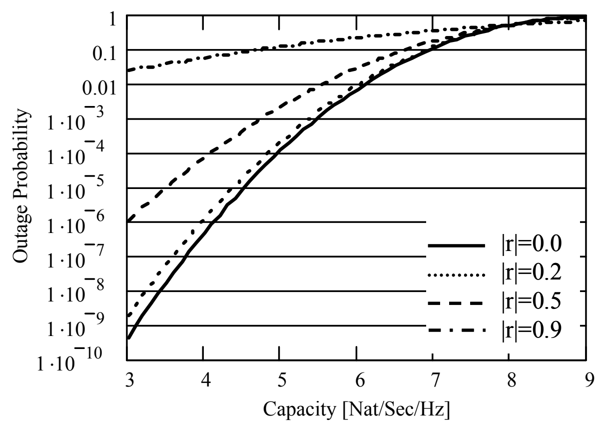

Fig. 4. Asymptotic outage capacity distributions of the 3x3 keyhole channel with exponential correlation.

However, for $|r| \leq 0.2$, $\sigma_c^2 \approx \sigma_u^2$. In this case the correlation has no significant impact on the asymptotic capacity unless a very low outage probability is of interest. When $P_{out} > 0.5$, the corresponding outage capacity of the correlated channel is larger than that of the uncorrelated one. However, this range of probabilities has little importance if any from the practical point of view. We show later on that the exact outage capacity distribution of a 3x3 keyhole channel is already close to the asymptotic one. To demonstrate the effect of correlation discussed above, Fig. 4 shows the asymptotic outage capacity distributions of the 3x3 keyhole channels with exponential correlation at both Tx and Rx ends for $\gamma_0 = 30dB$. We see, in particular, that the distributions for $|r| = 0$ and $|r| = 0.2$ are almost the same for $P_{out} \geq 10^{-9}$, however, as $|r|$ increases the outage capacity decreases drastically at low outage probabilities ($P_{out} < 0.1$).

Another physically-based single-parameter correlation matrix model is the quadratic exponential (QE) one. In this model the elements of the correlation matrix $\mathbf{R}$ are given as [14]:

$$R_{km} = \begin{cases} r^{(m-k)^2}; & m \geq k \\ \bar{r}^{(k-m)^2}; & m < k \end{cases}, \quad |r| < 1 \tag{24}$$

Here the correlation between different antennas decays much faster with distance than in the previous example. This model describes well the scenario with Gaussian angular spectrum. Unlike the previous model, there are such complex-valued r for which $\mathbf{R}$ is not positive semi-definite. These values should be avoided. Apparently for the QE correlation matrix $\lim_{n\to\infty} n^{-1} trace\{\mathbf{R}\} = 1$ and $\lim_{n\to\infty} n^{-2} \|\mathbf{R}\|^2$ can be bounded from above and below using the Cauchy Integral formula as follows:

$$\lim_{n\to\infty} n^{-2} \|\mathbf{R}\|^2 \leq \lim_{n\to\infty} \frac{1}{n} \cdot \left[1 + \sqrt{\frac{\pi}{-2\ln|r|}}\right] \tag{25}$$

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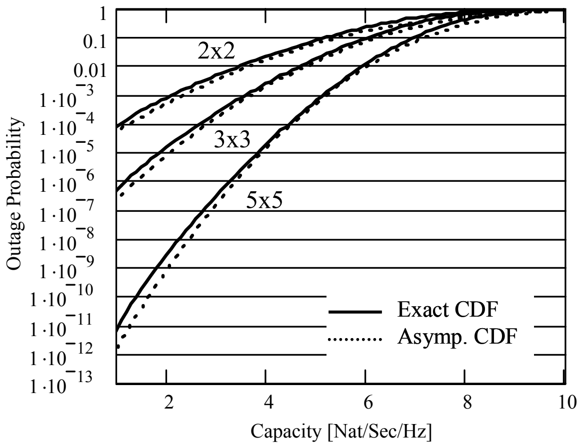

Fig. 5. Keyhole channel outage capacity: exact vs. asymptotic distribution.

$$\lim_{n\to\infty} n^{-2}\|\mathbf{R}\|^2 \ge \lim_{n\to\infty}\frac{1}{n}\cdot\left[1+\sqrt{\frac{\pi}{-2\ln|r|}}\cdot erfc\{\sqrt{-2\ln|r|}\}\right] \quad (26)$$

where $erfc\{x\}$ is the complimentary error function. Since $\lim_{n\to\infty} n^{-2}\|\mathbf{R}\|^2 = 0$, from Theorem 1 the asymptotic outage capacity distribution of the keyhole channel with the QE correlation at both Tx and Rx ends is Gaussian with the mean given in (19) and the variance bounded by either (25) or (26). We notice that using the upper-bound (25) in place of the variance of the asymptotic outage capacity distribution makes the latter closer to the exact distribution when n_t and n_r are finite. Thus we limit further discussion to the upper bound only.

Similarly to the previous analysis, assuming that $n = n_t = n_r$ and $r = r_t = r_r$, the asymptotic capacity distributions of an uncorrelated $r = 0$ and correlated $r \neq 0$ keyhole channels have the same mean $\ln(1 + n\gamma_0)$ and the variances ratio η'' can be estimated using the upper bound in (25):

$$\eta'' \overset{\Delta}{=} \frac{\sigma_c^2}{\sigma_u^2} = 1 + \sqrt{\frac{\pi}{-2\ln|r|}},\ |r| < 1 \quad (27)$$

For any $|r| < 1$, $\eta'' \ge 1$ and it is a monotonically increasing function of $|r|$. Therefore the effect of correlation in this case is similar to that of the exponential correlation matrix. The outage capacity decreases with increasing $|r|$ for $P_{out} < 0.5$.

While the Gaussian approximation for the outage capacity distribution is simple, it is valid when both n_t and n_r tend to infinity. For given n_t and n_r, the exact distribution approaches the asymptotic one faster for larger average SNR. Moreover, in some cases, the asymptotic distribution follows closely the exact one starting from 2x2 keyhole channel. To demonstrate this, the exact and asymptotic capacity distributions of 2x2, 3x3 and 5x5 keyhole MIMO channels with the exponential correlation are shown in Fig. 5. The correlation parameters are $|r_t| = |r_t| = 0.7$ and $\gamma_0 = 30dB$. Clearly, the difference is negligible for most practical purposes. However, there are cases when the discrepancy between the exact and asymptotic outage capacity distributions is larger. We explain this by the slow convergence of $n_t^{-1}\beta_t$ and $n_t^{-1}\beta_r$ to the corresponding Gaussian random variables.

V. Conclusion

There is a similarity between the asymptotic outage capacity distribution of the spatially correlated keyhole channel and that of the uncorrelated [11] and correlated [12] Rayleigh MIMO channels, which turn to be Gaussian in all considered cases. The fact that the outage capacities of so different channels are all asymptotically Gaussian may be an indication of generality of the Gaussian model in the asymptotic outage capacity analysis of MIMO systems.

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This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the WCNC 2006 proceedings.