Analytical BER Analysis of the V-BLAST in a Rayleigh Fading Channel

Sergey Loyka
School of Information Technology and Engineering,
University of Ottawa,
161 Louis Pasteur, Ottawa, Ontario, Canada, K1N 6N5
E-mail: sergey.loyka@ieee.org

Francois Gagnon
Department of Electrical Engineering
Ecole de Technologie Superieure
1100, Notre-Dame St. West, Montreal
Quebec, H3C 1K3, Canada
E-mail: francois.gagnon@etsmtl.ca

Abstract— The BLAST algorithm is simple and, hence, popular solution for a signal processing at the MIMO receiver. In this paper, we present a closed-form analytical analysis of the V-BLAST without optimal ordering. A result on the zero-forcing maximum ratio combining weights at each detection step is derived to obtain a number of results: independence of noise, distribution of signal to noise ratio and block or bit error rates. We present a detailed analytical analysis and closed-form expressions for instantaneous and average BER at each detection step, which account for the error propagation and hold true for any modulation format and take simple form in some cases (BPSK). Asymptotic form, for large average SNR, of these expressions is especially simple; the effect of the error propagation in this mode is to increase the total average BER by about 20, which is not catastrophic at all. It is demonstrated that the conventional V-BLAST and QR-decomposition based V-BLAST are essentially identical and have the same performance. Extensive Monte-Carlo simulations validate the analytical results and conclusions.

Index Terms—MIMO, V-BLAST, multi-antenna system, BER, outage, fading,

I. INTRODUCTION

The BLAST algorithm is simple and, hence, popular solution for a signal processing at the MIMO receiver, which also achieves the full MIMO capacity [1-3]. Its BER performance has been studied mainly numerically (Monte-Carlo techniques) since analytical analysis presents serious difficulties, especially when no bounds or approximations are used. While 2xn system (i.e. with 2 Tx and n Rx antennas) can be analytically analyzed in a closed form without using any bounds [4,5], an extension of the analysis to the general case of mxn system has not been found yet. Hence various bounds and approximations have been employed to attack the problem [6]. Consequently, the solutions found are limited in some way. A limitation of [4,5] is that a non-coherent equal gain combining (NC EGC) was used after interference nulling out (projection), which is not optimum, and that the after-projection noise correlation was ignored.

The approach adopted in the present paper is different. Close examination of the problem of BLAST BER performance analysis reveals that the major difficulty for closed-from exact analytical analysis is due to the optimal ordering procedure. Hence, we analyze the algorithm performance without optimal ordering. Clearly, this is a disadvantage of the analysis. However, there are certain advantages as well: (i) closed-form exact analytical analysis is possible in the general case of mxn system, which results in exact BER expressions, (ii) this provides deep insight and understanding that cannot be gained using Monte-Carlo approach alone, (iii) there exists a hope that the techniques developed can be further extended to account for the optimal ordering, (iv) comparing the performance of the no-ordering algorithm to that with the optimal ordering allows one to better understand the advantages provided by the ordering and various differences in the performance, (v) computational complexity of the algorithm without optimal ordering is significantly less and, hence, an implementation complexity is smaller. Contrary to [5], we employ the maximum ratio combining (MRC) after the interference projection (taking into account the after-projection noise correlation), which provides the best performance in terms of output SNR and, hence, BER. This also results in exact closed-form expressions for unconditional average BER at each step, which is not available in [5], or, to the best of our knowledge, elsewhere.

The approach consists essentially in obtaining new results on the optimal MRC weights without ordering. Theses weights are expressed as a product of a projection matrix by part of the channel matrix and it is thus possible to prove that the after-combining noise and inter-stream interference (ISI) at each detection step are Gaussian and independent. A closed form expression of the output SNR distributions at each step is obtained with an assumption of error-free detection at the previous steps. These distributions are used to obtain relatively precise outage, block, bit and total bit error rates. A more detailed analysis, which still makes good use of the MRC weight properties but with conditioning on previous step error events, permits to obtain exact BER expressions for BPSK modulation without the error-free detection assumption, i.e. taking into account the error propagation. Finally, by analyzing the details of the QR decomposition and V-BLAST, it is demonstrated that the MRC weight matrix and the Q matrix in QR decomposition are equal and, hence, performance of the regular V-BLAST and the QR- based BLAST are the same.

II. THE V-BLAST ALGORITHM

The main idea of the V-BLAST is to split the information bit stream into several sub-streams and transmit them in parallel using a set of transmit (Tx) antennas at the same time and frequency. At the receiver (Rx) end, each Rx antennas "sees" all the transmitted signals, which are mixed due to the nature

of the wireless propagation channel. Using appropriate signal processing at the Rx, these signals can be unmixed so that the matrix wireless channel is transformed into a set of virtual parallel independent channels (provided that the multipath is rich enough).

The standard baseband system model is given by

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{\xi} \qquad (1)$$

where $\mathbf{s}$ and $\mathbf{r}$ are the Tx and Rx vectors correspondingly, $\mathbf{H}$ is the mxn channel matrix, i.e. the matrix of the complex channel gains between each Tx and each Rx antenna, n is the number of Rx antennas, m is the number of Tx antennas, $n \geq m$, and $\mathbf{\xi}$ is the additive white Gaussian noise (AWGN), which is assumed to be $\mathcal{CN}(0, \sigma_0^2 \mathbf{I})$, i.e. independent and identically distributed (i.i.d.) in each branch.

The job of the V-BLAST algorithm is to find $\mathbf{s}$ given $\mathbf{r}$ and $\mathbf{H}$ in a computationally-efficient way. The V-BLAST processing begins with the 1st Tx symbol and proceeds in sequence to the m-th symbol. When the optimal ordering procedure is employed, the Tx indexing is changed prior to the processing. The main steps of the algorithm are as follows [1,3]:

(1) *The interference cancellation step*: at the i-th processing step (i.e., when the signal from the i-th transmitter is detected) the interference from the first (i-1) transmitters can be subtracted based on the estimations of the Tx symbols and the knowledge of the channel matrix $\mathbf{H}$,

$$\mathbf{r}_i' = \mathbf{r} - \sum_{j=1}^{i-1} \mathbf{h}_j \hat{s}_j \qquad (2)$$

where $\mathbf{h}_j$ is the j-th column of $\mathbf{H}$, and $\hat{s}_j$ are the detected symbols (which are assumed to be error-free).

(2) *The interference nulling step*: based on the knowledge of the channel matrix, the interference from yet-to-be-detected symbols can be nulled out using the Gram-Schmidt orthogonalization process (applied to the column vectors of $\mathbf{H}$) and orthogonal projection on the sub-spaced spanned by yet-to-be-detected symbols,

$$\mathbf{r}_i'' = \mathbf{P}_i \mathbf{r}_i' \qquad (3)$$

where $\mathbf{P}_i$ is the projection matrix on the sub-space orthogonal to that spanned by $\{\mathbf{h}_{i+1}\mathbf{h}_{i+2}...\mathbf{h}_m\}$: $\mathbf{P}_i = \mathbf{I} - \mathbf{H}_i(\mathbf{H}_i^+ \mathbf{H}_i)^{-1}\mathbf{H}_i^+$, where $\mathbf{H}_i = [\mathbf{h}_{i+1}\mathbf{h}_{i+2}...\mathbf{h}_m]$ [8].

(3) *The optimal ordering procedure*: the order of symbol processing is organized according to their after-processing SNRs in the decreasing order, i.e. the symbol with highest SNR is detected first.

III. ANALYSIS OF THE V-BLAST ALGORITHM

The following basic assumptions are employed in the present paper:

(1) The channel is random, quasistatic (i.e. fixed for every frame of information bits but varying from frame to frame), frequency independent (i.e., negligible delay spread); the components of $\mathbf{H}$ are $\mathcal{CN}(0, \mathbf{I})$ (i.e., i.i.d. Rayleigh fading with unit average power gain).

(2) Equal-power constellations are used.

(3) The Tx signals, noise and channel gains are independent of each other.

(4) Perfect channel knowledge is available at the receiver, but not at the transmitter.

(5) There is no performance degradation due to synchronization and timing errors.

It should be noted that some of the results below do not need all these assumptions. As it was indicated above, the optimal ordering procedure will be omitted in the present paper. We follow the approach to V-BLAST analysis proposed in [4,5], where it was shown that the conditional (i.e. assuming no detection error at the first (i-1) steps) after-processing instantaneous (i.e. for given channel instant) signal power P_{si} at i-th processing step is

$$P_{si} \sim \chi_{2(n-m+i)}^2, \qquad (4)$$

where $\sim$ means equal in distribution, and different P_{si} are independent of each other. The i-th step has diversity order equal to $(n-m+i)$, the smallest one being at the 1st step and the largest – at the last one. Note that the fact that the distribution is conditional (no error propagation) does not limit the analysis since, as we show below, the conditional distribution at each step is sufficient to find the block error rate (BLER) and outage probability taking into account the error propagation. The distribution of P_{si} follows also from the Bartlett decomposition of the complex Wishart matrix [7].

The best way to improve the output SNR is to use maximum ratio combining (MRC) after the interference nulling out step, i.e. to form the decision variable

$$\hat{r}_i = \mathbf{w}_i^+ \mathbf{r}_i'' , \qquad (5)$$

where $\mathbf{w}_i^+$ are the optimum weights, and "+" means Hermitian conjugate. However, the well-known MRC expressions for the weights and the output SNR cannot be applied directly since the orthogonal projection during the nulling-out step (see (3)) results in correlated branch noise. The after-projection noise correlation matrix is

$$\mathbf{C}_{\xi i} = \left\langle \mathbf{P}_i \mathbf{\xi}\mathbf{\xi}^+ \mathbf{P}_i \right\rangle = \sigma_0^2 \mathbf{P}_i , \qquad (6)$$

where $< >$ denotes expectation (in this case, over the noise). The MRC combining weights $\mathbf{w}_i$ are given in this case by the solution of the following generalized eigenvalue problem,

$$\left(\mathbf{C}_{si} - \gamma_i \mathbf{C}_{\xi i} \right) \mathbf{w}_i = 0 , \qquad (7)$$

where $\mathbf{C}_{si}$ is the after-projection signal covariance matrix at step *i*. As detailed analysis demonstrates (the proof is omitted here due to the page limits; more details are given in the extended journal version of this paper [13]), the optimum (MRC) weights are given by

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$$\mathbf{w}_i = \mathbf{P}_i \mathbf{h}_i \,, \qquad (8)$$

Since these weights already include the projection, they can be applied directly to $\mathbf{r}_i'$ (i.e. these are zero-forcing MRC weights). The output signal can be presented as

$$\hat{r}_i = \mathbf{w}_i^+ \mathbf{r}_i' = r_{si} + r_{\xi i}, \; r_{si} = \mathbf{w}_i^+ \mathbf{h}_i s_i, \; r_{\xi i} = \mathbf{w}_i^+ \boldsymbol{\xi} \,, \qquad (9)$$

Remarkably, the output SNR is still the same as that for an i.i.d. branch noise,

$$\gamma_i = \sigma_0^{-2} P_{si} = \sigma_0^{-2} \mathbf{y}_i^+ \mathbf{y}_i = \sigma_0^{-2} \mathbf{h}_i^+ \mathbf{P}_i \mathbf{h}_i \,, \qquad (10)$$

where $\mathbf{y}_i = \mathbf{P}_i \mathbf{h}_i s_i$ is the after-projection signal and we have assumed, without loss of generality, that $|s_i|^2 = 1$ (i.e., equal power constellation). We stress that this is a non-trivial result that holds true because of special structure of the projection matrix $\mathbf{P}$.

Since P_{si} at different steps are independent of each other, so are γ_i. From this, however, it does not follow that the decisions and, hence, errors at each step are independent of each other as the latter requires for the noise $r_{\xi i}$ to be independent at each step, not just the SNR. It can be shown (based on the properties of the projection matrices and using (8)) that the optimum weights at different steps are orthogonal to each other,

$$\mathbf{w}_i^+ \mathbf{w}_j = \mathbf{h}_i^+ \mathbf{P}_i \mathbf{P}_j \mathbf{h}_j = \mathbf{0} \; \forall i \neq j \,, \qquad (11)$$

Consider now the output noise covariance matrix,

$$R_{ij} = \langle r_{\xi i} r_{\xi j} \rangle \,, \qquad (12)$$

where the expectation is over the noise. Using (8), (9) and (11), one obtains,

$$R_{ij} = \sigma_0^2 \mathbf{w}_i^+ \mathbf{w}_j = \sigma_0^2 |\mathbf{w}_i|^2 \delta_{ij} \,, \qquad (13)$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Since $r_{\xi i} = \mathbf{w}_i^+ \boldsymbol{\xi}$ are complex Gaussian (for given channel realization), independence follows from zero correlation. Hence, the decisions and errors are independent at each step and the conditional error rates are independent too.

IV. EXACT BER EXPRESSIONS FOR BPSK MODULATION

In this section, we derive exact closed-form expressions for step BER with BPSK modulation. We use the normalized weights $|\mathbf{w}_i|^2 = 1$ (note that normalization does not affect SNR and, hence, BER) so that the weight matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, ... \mathbf{w}_m]$ is semi-unitary (it is unitary only when $m = n$),

$$\mathbf{W}^+ \mathbf{W} = \mathbf{I} \qquad (14)$$

This normalization corresponds to $\mathbf{w}_i = \mathbf{P}_i \mathbf{h}_i / |\mathbf{P}_i \mathbf{h}_i|$. For simplicity, we begin with 2xn system. In our analysis, we follow a similar approach of multiuser detection BER analysis [11].

A. 2xn V-BLAST BER

The output signal at step 2 is

$$\hat{r}_2 = \mathbf{w}_2^+ \mathbf{h}_2 s_2 + \mathbf{w}_2^+ \mathbf{h}_1 \Delta s_1 + \mathbf{w}_2^+ \boldsymbol{\xi} \,, \qquad (15)$$

where $\mathbf{w}_2 = \mathbf{h}_2 / |\mathbf{h}_2|$ and $\Delta s_1 = s_1 - \hat{s}_1$. Conditioned on $\mathbf{h}_2$ and Δs_1, the inter-stream interference (ISI) $z_{21} = \mathbf{w}_2^+ \mathbf{h}_1 \Delta s_1$ is Gaussian, $z_{21}|_{\mathbf{h}_2, \Delta s_1} \sim CN(0, |\Delta s_1|^2)$. To see this, we first note that, for given $\mathbf{h}_2$ and Δs_1, z_{21} is a sum of Gaussian random variables, and, hence, is Gaussian. The conditional mean and variance of z_{21} are

$$\langle z_{21} \rangle = 0, \; \langle |z_{21}|^2 \rangle = \mathbf{w}_2^+ \langle \mathbf{h}_1 \mathbf{h}_1^+ \rangle \mathbf{w}_2 |\Delta s_1|^2 = |\Delta s_1|^2, \quad (16)$$

as $\langle \mathbf{h}_1 \rangle = \mathbf{0}$ and $\langle \mathbf{h}_1 \mathbf{h}_1^+ \rangle = \mathbf{I}$. Since the conditional distribution does not depend on $\mathbf{h}_2$, the conditioning on $\mathbf{h}_2$ can be dropped, $z_{21}|_{\Delta s_1} \sim CN(0, |\Delta s_1|^2)$. In the same way, one can see that the noise term in (15) is also Gaussian and independent of $\mathbf{h}_2$, $\mathbf{w}_2^+ \boldsymbol{\xi} \sim CN(0, \sigma_0^2)$, and it is also independent of the other terms. The last two terms can be joined to get the "total noise" term, which includes the ISI, and, conditioned on Δs_1, its distribution is

$$\mathbf{w}_2^+ \mathbf{h}_1 \Delta s_1 + \mathbf{w}_2^+ \boldsymbol{\xi} \sim CN(0, \sigma_0^2 + |\Delta s_1|^2) \,, \qquad (17)$$

Since instantaneous BERs at each step are independent of each other and $|\Delta s_1| = 0$ (no error at step 1) with probability $1 - Q(\sqrt{2\gamma_1})$ and $|\Delta s_1| = 2$ (error at step 1) with probability $Q(\sqrt{2\gamma_1})$, for given $\mathbf{h}_2$ BER at step 2 can be immediately found from (15) as

$$P_{u2}(\mathbf{h}_2) = Q\left(\sqrt{2\gamma_2}\right)\left(1 - \overline{P}_{e1}\right) + Q\left(\sqrt{\frac{2\mathbf{h}_2^+ \mathbf{h}_2}{\sigma_0^2 + 4}}\right)\overline{P}_{e1} \,, \; (18)$$

where $\overline{P}_{e1} = \langle Q(\sqrt{2\gamma_1}) \rangle_{\gamma_1}$ is the average BER at step 1 (which is the same as (n-1)-order MRC average BER) and, with the adopted normalization, $\gamma_0 = 1/\sigma_0^2$ is the average SNR. $P_{u2}(\mathbf{h}_2)$ is unconditional (including error propagation from step 1) BER at step 2 averaged over $\mathbf{h}_1$ but not $\mathbf{h}_2$ (it is neither average not instantaneous BER in traditional sense). Finally, the average unconditional BER at step 2 is

$$\overline{P}_{u2} = \langle P_{u2}(\mathbf{h}_2) \rangle_{\mathbf{h}_2} = \overline{P}_{e2}\left(1 - \overline{P}_{e1}\right) + \overline{P}_{21} \overline{P}_{e1} \,, \quad (19)$$

where $\overline{P}_{e2} = \langle Q(\sqrt{2\gamma_2}) \rangle_{\gamma_2}$ is the conditional (on no error at 1st step) average BER at step 2 (which is the same as n-th order MRC average BER), and

$$\overline{P}_{21} = \left\langle Q\left(\sqrt{\frac{2\mathbf{h}_2^+ \mathbf{h}_2}{\sigma_0^2 + 4}}\right) \right\rangle_{\mathbf{h}_2} = \overline{P}_{e2}\left(\frac{1}{\sigma_0^2 + 4}\right), \quad (20)$$

is the average probability of error propagation. In the case of 2x2 system the expressions become especially simple,

$$\overline{P}_{e1} = (1 - \mu)/2 \,, \; \overline{P}_{e2} = (1 - \mu)^2 (2 + \mu)/4 \,, \; (21)$$

where $\mu = \sqrt{\gamma_0/(1+\gamma_0)}$. For large average SNR, $\sigma_0^2 \ll 1$,

$$\overline{P}_{21} = \overline{P}_2\left(\frac{1}{\sigma_0^2+4}\right) \approx \frac{1}{5}, \quad \overline{P}_2 \approx \frac{3}{4\gamma_0^2}, \quad \overline{P}_1 \approx \frac{1}{4\gamma_0}, \quad (22)$$

and (19) simplifies to

$$\overline{P}_{u2} \approx \overline{P}_{e2} + \overline{P}_{21}\overline{P}_{e1} \approx \overline{P}_{e1}/5 \approx 1/(20\gamma_0), \quad (23)$$

Note that the second term dominates in (19) as it is the 1st order term (no diversity) while the 1st term is the 2nd order one (2nd order diversity). Clearly, there is no diversity at the 2nd step due to the error propagation. The total average BER (i.e. the BER in the single output stream, which includes all the individual sub-streams) can be found as

$$\overline{P}_{et} = \frac{1}{m}\sum_{i=1}^{m}\overline{P}_{ui} \quad (24)$$

In the large SNR case, this reduces to

$$\overline{P}_{et} \approx \overline{P}_{e1}/2 \approx 1/8\gamma_0 \quad (25)$$

as the first step BER is dominant. Additionally, block error rate (BLER) can be defined as a probability of having at least one error at the Tx vector, which is also a useful performance measure. The average BLER is

$$\overline{P}_B = \overline{P}_{e2}\left(1-\overline{P}_{e1}\right) + \overline{P}_{e1}, \quad (26)$$

which can be further approximated, for high average SNR, as

$$\overline{P}_B \approx \overline{P}_{e1} \approx 1/(4\gamma_0), \quad (27)$$

i.e. the first step average BER dominates in the average BLER, which provides a quick way to estimate the latter.

B. nxm V-BLAST BER

The derivation in this case follows the same steps as above and is omitted here due to the page limits. The results are as follows. The conditional BER at step i averaged over $\{\mathbf{h}_1, \mathbf{h}_2, ... \mathbf{h}_{i-1}\}$ is

$$P_{e,i|\mathbf{E}_{i-1}} = \Pr\{e_i = 1 | \mathbf{E}_{i-1}\} = Q\left(\sqrt{\frac{2\mathbf{h}_i^+ \mathbf{P}_i \mathbf{h}_i}{\sigma_0^2 + 4|\mathbf{E}_{i-1}|^2}}\right), \quad (28)$$

where $E_{i-1} = [e_1, e_2 ... e_{i-1}]$ is the error vector: $e_k = 1$ if there is an error at step k and $e_k = 0$ otherwise. $\Pr\{\mathbf{E}_{i-1}\}$ can be expressed as

$$\Pr\{\mathbf{E}_{i-1}\} = \prod_{k=1}^{i-1}\Pr\{e_k|\mathbf{E}_{k-1}\}, \quad (29)$$

where

$$\Pr\{e_k|\mathbf{E}_{k-1}\} = \begin{cases} P_{e,k|\mathbf{E}_{k-1}}, & e_k = 1 \\ 1 - P_{e,k|\mathbf{E}_{k-1}}, & e_k = 0 \end{cases}, \quad (30)$$

The average unconditional BER at step i is

$$\overline{P}_{u,i} = \langle P_{u,i}(\mathbf{H})\rangle_{\mathbf{H}} = \sum_{\mathbf{E}_{i-1}} \overline{P}_{e,i|\mathbf{E}_{i-1}}\overline{P}[\mathbf{E}_{i-1}], \quad (31)$$

where the summation is over all possible error patterns $\mathbf{E}_{i-1}$ and

$$\overline{P}_{e,i|\mathbf{E}_{i-1}} = \left[\frac{1-\mu_i}{2}\right]^{n-m+i} \sum_{k=0}^{n-m+i-1} C_{n-m+i-1+k}^k \left[\frac{1+\mu_i}{2}\right]^k,$$

$$\mu_i = \left(\sqrt{1+\sigma_0^2 + 4|\mathbf{E}_{i-1}|^2}\right)^{-1}, \quad \overline{P}[\mathbf{E}_{i-1}] = \prod_{k=1}^{i-1}\langle\Pr\{e_k|\mathbf{E}_{k-1}\}\rangle$$

$C_n^k = n!/(k!(n-k)!)$ are the binomial coefficients, and $\langle\Pr\{e_k|\mathbf{E}_{k-1}\}\rangle$ are found using the expectation of (30) over appropriate entries of $\mathbf{H}$. Using (31) and (24), the total average BER can be easily obtained.

V. EQUIVALENCE OF QR AND REGULAR V-BLAST

Using an QR decomposition instead of the interference nulling procedure of the traditional V-BLAST algorithm has been proposed in [12] to reduce computational complexity of the algorithm. In this section, we prove that the QR-decomposition based BLAST is identical to the regular (i.e., using projections to null out the inter-stream interference) V-BLAST both in terms of performance and basic processing steps. In fact, the regular BLAST does perform QR decomposition of the channel matrix in an implicit way.

The QR based BLAST operates as follows. Consider a QR decomposition of the channel matrix,

$$\mathbf{H} = \mathbf{QR}, \quad (32)$$

where $\mathbf{Q}$ is semi-unitary, $\mathbf{Q}^+\mathbf{Q} = \mathbf{I}$, and $\mathbf{R}$ is lower-triangular, $r_{ij} = 0$ for $j > i$. Left-multiplying (1) by $\mathbf{Q}^+$, one obtains,

$$\mathbf{r}' = \mathbf{Qr} = \mathbf{Rs} + \xi', \quad (33)$$

where $\xi' = \mathbf{Q}\xi$. Since $\mathbf{R}$ is lower-triangular, the interference from yet to be detected symbols is eliminated,

$$r_i' = \sum_{j=1}^{i} r_{ij}s_j + \xi_i', \quad (34)$$

On the other hand, consider the regular V-BLAST. The normalized weight matrix $\mathbf{W}$ is semi-unitary, and, hence, left-multiplying $\mathbf{H}$ by $\mathbf{W}^+$ yields

$$\mathbf{W}^+\mathbf{H} = \mathbf{T}, \quad (35)$$

where $\mathbf{T}$ is lower-triangular. This is also a QR decomposition of $\mathbf{H}$. Since QR decomposition is unique [7] (up to an ordering), we conclude that $\mathbf{W} = \mathbf{Q}$ and $\mathbf{R} = \mathbf{T}$ (if the same ordering is used in both cases). Hence, two algorithms essentially perform the same processing and, hence, the performance is the same. For an i.i.d. Rayleigh channel, the entries of $\mathbf{H}$ are i.i.d. complex Gaussian, $h_{ij} \sim CN(\mathbf{0}, \mathbf{I})$, and

the QR decomposition in (32) has the following properties [7]: 1) $r_{ii} \sim \chi^2_{2(n-m+i)}$, 2) $r_{ij} \sim CN(0,1)$, $i > j$, 3) all the elements of **R** are independent of each other, 4) the elements of **Q** and **R** are independently distributed and, hence, the noise in (33) is independent of the signal term **Rs**; furthermore, $\xi' \sim \xi \sim CN(\mathbf{0}, \sigma_0^2 \mathbf{I})$. These properties are identical to the results obtained above using detailed analysis from the signal processing perspective (the latter gives insight normally not available if the multivariate statistical results are adopted in the final form). This provides an additional validation of our results.

VI. Monte-Carlo Simulations

In order to verify the analytical results above, extensive Monte-Carlo (MC) simulations have been undertaken. Specifically, the Rayleigh i.i.d. fading channel and BPSK modulation demodulated coherently have been used. First, the instantaneous BER expressions have been validated. No statistically-significant difference between analytical and MC results have been found for conditional (without error propagation) and unconditional (with error propagation) BER. Secondly, the average BER expressions have been extensively validated. Some of the results are shown in Fig. 1-2.

Fig. 1 shows Monte-Carlo simulated average BER/BLER/TBER and the exact analytical results for 2x2 systems (for other system orders, similar results have been obtained and not shown here). Good agreement is obvious. 1st step BER dominates the BLER for high SNR (>5 dB). Note that the error propagation has significant effect on the 2nd step BER, as comparison to the "no error propagation" BER demonstrates (as (23) indicates, the error propagation results in the 2nd step diversity order being equal to one rather than two).

Comparison of the exact and approximate average BLER/TBER expressions of the 2x2 system indicates that the average BLER is well approximated by the 1st step average BER for $\gamma_0 \geq 5dB$ (i.e. eq. 27). The accuracy of the TBER approximation (eq. 25) is a bit worse but still acceptable. The small inaccuracy is due to the fact that (25) ignores the error propagation, which demonstrates that the effect of error propagation on the total average BER is indeed small. Including it results in a better approximation,

$$\overline{P}_{et} \approx \overline{P}_{e1}\left(1 + \overline{P}_{21}\right)/2 \approx 1/(7\gamma_0) \qquad (36)$$

Comparing (36) to (25), which ignores the error propagation, one concludes that the effect of error propagation on the average total BER is to increase it by about 20%, which is indeed small (contrary to a widely-accepted viewpoint that the error propagation has a dramatically negative effect on V-BLAST BER). On the contrary, the 2nd step average BER is affected by the error propagation in a dramatic way: the diversity order decreases from 2 without error propagation to 1 with it (see Fig. 1). Fortunately, since the 1st step BER is dominant, this does not affect significantly the total BER.

Fig. 2 gives the average step BER of 4x4 system along with the average BLER/TBER. Based on these numerical
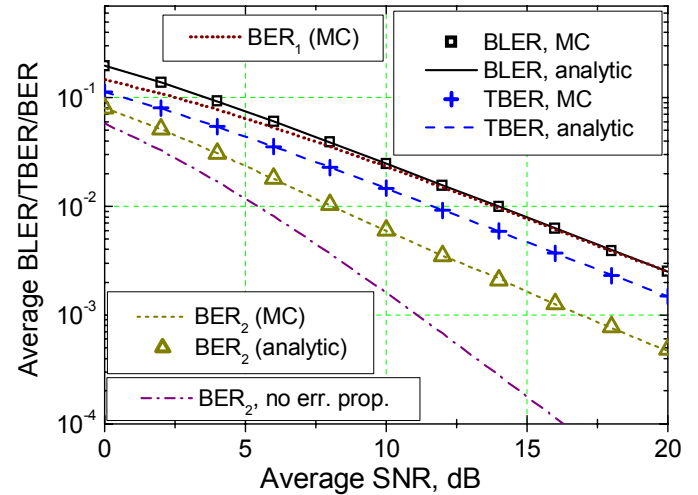

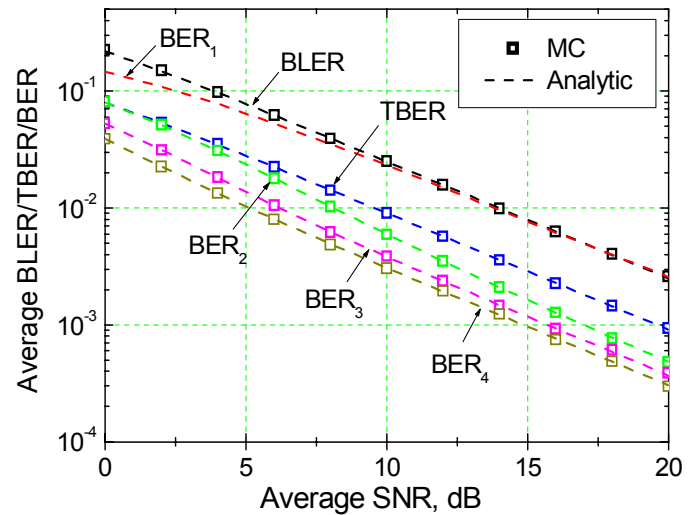Fig. 1. Average BLER/TBER/BER of 2x2 V-BLAST

Fig. 2. Average BLER/TBER/BER of 4x4 V-BLAST

results, one observes that the average step BER, despite of the error propagation, decreases with step number,

$$\overline{P}_{u(i+1)} < \overline{P}_{ui} \qquad (37)$$

According to the analysis above, we attribute this to the fact that (i) the probability of error propagation is less than 1 ($\approx 1/5$ for 2x2 system) and (ii) the conditional BER at higher steps has higher diversity order (see (22) for the case of 2x2 system). Clearly, the average step BER of nxn system can be well approximated by

$$\overline{P}_{ui} \approx a_i / \gamma_0 \qquad (38)$$

where a_i are some constants, $a_{i+1} < a_i$. We conjecture that $\lim_{i \to \infty} a_i = 0$. The approximation in (38) also applies to BLER and TBER. Additionally, one observes that, for high

SNR, $\overline{P}_{et} \approx \overline{P}_B / m \approx \overline{P}_{e1} / m$. Based on this, we conjecture that $\lim_{n \to \infty} \overline{P}_{et} = 0$ for nxn system, which demonstrates that V-BLAST is asymptotically optimum space-time architecture that achieves the MIMO capacity at arbitrary low BER, even without optimal ordering. Comparing Fig. 2 to Fig. 1, one concludes that the average BER at given step does not depend on whether higher steps are presents or not (for example, $\overline{P}_{e1}$ is the same for 2x2, 3x3 and 4x4 systems), which also follows from (31). This conclusion is not entirely trivial as step 1 processing implies projecting out the sub-space spanned by $[\mathbf{h}_2 \mathbf{h}_3 ... \mathbf{h}_m]$ and, hence, depends on the presence of higher-order steps.

VII. REFERENCES

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