ELG6108: Introduction to Convex Optimization

Lecture 4: Convex optimization problems

Dr. Sergey Loyka

EECS, University of Ottawa

March 8, 2021

Convex optimization problems[1]

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- robust optimization
- geometric programming
- semidefinite programming (rate of MIMO channel, secrecy rate)

[1] adapted from Boyd & Vandenberghe, Convex Optimization, Lecture slides.

Optimization problem in standard form

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\
\text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\
& h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p
\end{aligned}
\tag{1}
$$

- $\mathbf{x}$ is the optimization variable (vector)
- f_0 : is the objective or cost function
- f_i: are the inequality constraint functions
- h_i: are the equality constraint functions

Optimization problem in standard form

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{aligned} \tag{1}$$

- $\mathbf{x}$ is the optimization variable (vector)
- f_0 : is the objective or cost function
- f_i: are the inequality constraint functions
- h_i: are the equality constraint functions

optimal value:

$$p^\star = \min_{\mathbf{x}} \{ f_0(\mathbf{x}) : f_i(\mathbf{x}) \leq 0, h_i(\mathbf{x}) = 0 \ \forall i \} \tag{2}$$

Optimization problem in standard form

$$\begin{array}{ll} \min_{\mathbf{x}} & f_0(\mathbf{x}) \\ \text{s.t.} & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{array} \tag{1}$$

- $\mathbf{x}$ is the optimization variable (vector)
- f_0 : is the objective or cost function
- f_i: are the inequality constraint functions
- h_i: are the equality constraint functions

optimal value:

$$p^\star = \min_{\mathbf{x}} \{ f_0(\mathbf{x}) : f_i(\mathbf{x}) \leq 0, h_i(\mathbf{x}) = 0 \ \forall i \} \tag{2}$$

- $p^\star = \infty$ if problem is infeasible (no $\mathbf{x}$ satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

(globally) Optimal and locally optimal points

- $\mathbf{x}$ is **feasible** if $\mathbf{x} \in \text{dom } f_0$ and it satisfies the constraints

(globally) Optimal and locally optimal points

- $\mathbf{x}$ is **feasible** if $\mathbf{x} \in \mathrm{dom}\, f_0$ and it satisfies the constraints
- a feasible $\mathbf{x}$ is **optimal** if $f_0(\mathbf{x}) = p^\star$

(globally) Optimal and locally optimal points

- $\mathbf{x}$ is **feasible** if $\mathbf{x} \in \text{dom } f_0$ and it satisfies the constraints
- a feasible $\mathbf{x}$ is **optimal** if $f_0(\mathbf{x}) = p^\star$
- $\mathbf{x}$ is **locally optimal** if there is an $d > 0$ such that $\mathbf{x}$ is optimal for

$$\begin{aligned} \min_{\mathbf{z}} \quad & f_0(\mathbf{z}) \\ \text{s.t.} \quad & f_i(\mathbf{z}) \le 0, \quad h_i(\mathbf{z}) = 0 \ \forall \ i \\ & \|\mathbf{z} - \mathbf{x}\|_2 \le d \end{aligned} \tag{3}$$

Optimal and locally optimal points

examples (with $n = 1, m = p = 0$)

- $f_0(x) = 1/x,\ x > 0$: $p^\star = 0$, no optimal point

Optimal and locally optimal points

examples (with $n = 1, m = p = 0$)

- $f_0(x) = 1/x,\ x > 0$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x,\ x > 0$: $p^\star = -\infty$

Optimal and locally optimal points

examples (with $n = 1, m = p = 0$)

- $f_0(x) = 1/x,\ x > 0$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x,\ x > 0$: $p^\star = -\infty$
- $f_0(x) = x \log x,\ x > 0$: $p^\star = -1/e,\ x = 1/e$ is optimal

Optimal and locally optimal points

examples (with $n = 1, m = p = 0$)

- $f_0(x) = 1/x,\ x > 0$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x,\ x > 0$: $p^\star = -\infty$
- $f_0(x) = x \log x,\ x > 0$: $p^\star = -1/e,\ x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x,\ p^\star = -\infty$, local optimum at $x = 1$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$\mathbf{x} \in \mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i \tag{4}$$

- we call $\mathcal{D}$ the **domain** of the problem
- the constraints $f_i(\mathbf{x}) \leq 0, h_i(\mathbf{x}) = 0$ are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints $(m = p = 0)$

Implicit constraints

example:

$$\min \quad f_0(\mathbf{x}) = -\sum_{i=1}^{k} \log\left(\mathbf{b}_i - a_i^T \mathbf{x}\right) \tag{5}$$

is an unconstrained problem with implicit constraints $\mathbf{a}_i^T \mathbf{x} < b_i$

Feasibility problem

$$\begin{array}{ll} \text{find} & \mathbf{x} \\ \text{s.t.} & f_i(\mathbf{x}) \le 0, \quad i = 1, \dots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \dots, p \end{array} \tag{6}$$

can be considered a special case of the general problem with $f_0(\mathbf{x}) = 0$:

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & f_i(\mathbf{x}) \le 0, \quad i = 1, \dots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \dots, p \end{array} \tag{7}$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Convex optimization problem

Standard form convex optimization problem

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \le 0, \quad i = 1, \ldots, m \\ & \mathbf{a}_i^T \mathbf{x} = b_i, \quad i = 1, \ldots, p \end{aligned} \tag{8}$$

- $f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine

Convex optimization problem

Standard form convex optimization problem

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \le 0, \quad i = 1, \ldots, m \\ & \mathbf{a}_i^T \mathbf{x} = b_i, \quad i = 1, \ldots, p \end{aligned} \tag{8}$$

- $f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine
- problem is quasiconvex if f_0 is quasiconvex (and $f_1, \ldots, f_m$ convex)

Convex optimization problem

Standard form convex optimization problem

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & \mathbf{a}_i^T \mathbf{x} = b_i, \quad i = 1, \ldots, p \end{aligned} \tag{8}$$

- $f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine
- problem is quasiconvex if f_0 is quasiconvex (and $f_1, \ldots, f_m$ convex)

Another form of (8):

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{9}$$

Important property: feasible set is convex

Example

$$
\begin{aligned}
\min \quad & f_0(\mathbf{x}) = x_1^2 + x_2^2 \\
\text{s.t.} \quad & f_1(\mathbf{x}) = x_1 / \left(1 + x_2^2\right) \leq 0 \\
& h_1(\mathbf{x}) = (x_1 + x_2)^2 = 0
\end{aligned}
\tag{10}
$$

- convex problem?

Example

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(\mathbf{x}) = x_1 / \left(1 + x_2^2\right) \leq 0 \\ & h_1(\mathbf{x}) = (x_1 + x_2)^2 = 0 \end{aligned} \tag{10}$$

- convex problem?
- f_0 is convex; feasible set $\{(x_1, x_2) : x_1 = -x_2 \leq 0\}$ is convex

Example

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(\mathbf{x}) = x_1 / \left(1 + x_2^2\right) \leq 0 \\ & h_1(\mathbf{x}) = (x_1 + x_2)^2 = 0 \end{aligned} \tag{10}$$

- convex problem?
- f_0 is convex; feasible set $\{(x_1, x_2) : x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition)

Example

$$
\begin{aligned}
\min \quad & f_0(\mathbf{x}) = x_1^2 + x_2^2 \\
\text{s.t.} \quad & f_1(\mathbf{x}) = x_1 / \left(1 + x_2^2\right) \leq 0 \\
& h_1(\mathbf{x}) = (x_1 + x_2)^2 = 0
\end{aligned}
\tag{10}
$$

- convex problem?
- f_0 is convex; feasible set $\{(x_1, x_2) : x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition)
- equivalent (but not identical) to the convex problem

$$
\begin{aligned}
\min \quad & x_1^2 + x_2^2 \\
\text{s.t.} \quad & x_1 \leq 0 \\
& x_1 + x_2 = 0
\end{aligned}
\tag{11}
$$

- *Q: sketch the feasible set*

The **most important property**

- The **most important property** of a convex problem: any locally-optimal point is also globally-optimal

The **most important property**

- The **most important property** of a convex problem: any locally-optimal point is also globally-optimal

  which

- Makes it possible to solve convex problems *globally* and *efficiently*

The **most important property**

- The **most important property** of a convex problem: any locally-optimal point is also globally-optimal

  which

- Makes it possible to solve convex problems *globally* and *efficiently*

  and

- Does not hold for non-convex problems in general

  therefore

- "The great watershed in optimization is not between linearity and non-linearity, but convexity and non-convexity." - R.T. Rockafellar, 1993

Local and global optima

- The **most important property** of a convex problem: any locally-optimal point is also globally-optimal

Proof: by contradiction. Suppose $\mathbf{x}$ is locally optimal and $\mathbf{y}$ is globally-optimal with $f_0(\mathbf{y}) < f_0(\mathbf{x})$; $\mathbf{x}$ locally optimal means there is an $d > 0$ such that

$$\forall\, \mathbf{z} : \ \|\mathbf{z} - \mathbf{x}\|_2 \leq d \quad \Longrightarrow \quad f_0(\mathbf{z}) \geq f_0(\mathbf{x}) \tag{12}$$

Now consider $\mathbf{z} = \theta \mathbf{y} + (1 - \theta)\mathbf{x}$ with $\theta = d / \left(2\|\mathbf{y} - \mathbf{x}\|_2\right)$

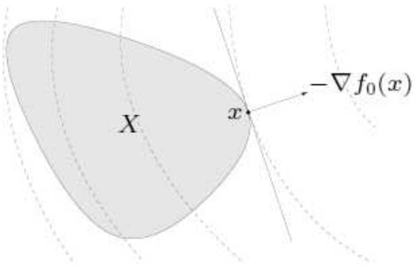
- $\|\mathbf{y} - \mathbf{x}\|_2 > d$, so $0 < \theta < 1/2$
- $\mathbf{z}$ is a convex combination of two feasible points, hence also feasible
- $\|\mathbf{z} - \mathbf{x}\|_2 = d/2$ and

$$f_0(\mathbf{z}) \leq \theta f_0(\mathbf{x}) + (1 - \theta) f_0(\mathbf{y}) < f_0(\mathbf{x}) \tag{13}$$

which contradicts the assumption that $\mathbf{x}$ is locally optimal!! Q.E.D.

Optimality criterion for differentiable f_0

$\mathbf{x}$ is optimal if and only if it is feasible and $\nabla f_0(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) \geq 0$ for all feasible $\mathbf{y}$



if nonzero, $\nabla f_0(\mathbf{x})$ defines a supporting hyperplane to feasible set X at $\mathbf{x}$

- **unconstrained problem:** min $f_0(\mathbf{x})$ $\rightarrow \mathbf{x} =$ optimal iff

$$\mathbf{x} \in \mathrm{dom}\, f_0, \quad \nabla f_0(\mathbf{x}) = 0 \tag{14}$$

- **unconstrained problem:** $\min \ f_0(\mathbf{x}) \ \rightarrow \mathbf{x} =$ optimal iff

$$\mathbf{x} \in \mathrm{dom}\, f_0, \quad \nabla f_0(\mathbf{x}) = 0 \tag{14}$$

- **minimization over nonnegative orthant**
  $\min \ f_0(\mathbf{x})$ s.t. $\mathbf{x} \succeq 0 \ \rightarrow \mathbf{x} =$ optimal iff

$$\mathbf{x} \in \mathrm{dom}\, f_0, \quad \mathbf{x} \succeq 0, \quad \begin{cases} \nabla f_0(\mathbf{x})_i \geq 0 & x_i = 0 \\ \nabla f_0(\mathbf{x})_i = 0 & x_i > 0 \end{cases} \tag{15}$$

- **unconstrained problem:** min $f_0(\mathbf{x}) \rightarrow \mathbf{x} =$ optimal iff

$$\mathbf{x} \in \text{dom } f_0, \quad \nabla f_0(\mathbf{x}) = 0 \tag{14}$$

- **minimization over nonnegative orthant**
  min $f_0(\mathbf{x})$ s.t. $\mathbf{x} \succeq 0 \rightarrow \mathbf{x} =$ optimal iff

$$\mathbf{x} \in \text{dom } f_0, \quad \mathbf{x} \succeq 0, \quad \begin{cases} \nabla f_0(\mathbf{x})_i \geq 0 & x_i = 0 \\ \nabla f_0(\mathbf{x})_i = 0 & x_i > 0 \end{cases} \tag{15}$$

- **equality constrained problem**

$$\min \ f_0(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \tag{16}$$

  $\mathbf{x}$ is optimal iff there exists a ν such that

$$\mathbf{x} \in \text{dom } f_0, \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \nabla f_0(\mathbf{x}) + \mathbf{A}^T \boldsymbol{\nu} = 0 \tag{17}$$

Equivalent Problems

Two problems are **equivalent** if a solution of one can be obtained from a solution of the other, and vice-versa

Some common transformations that preserve convexity:

- *eliminating equality constraints*

$$\min_{\mathbf{x}} \; f_0(\mathbf{x}) \quad \text{s.t. } f_i(\mathbf{x}) \leq 0, \; \mathbf{A}\mathbf{x} = \mathbf{b} \tag{18}$$

  is equivalent to

$$\min_{\mathbf{z}} \; f_0(\mathbf{F}\mathbf{z} + \mathbf{x}_0) \text{ s.t. } f_i(\mathbf{F}\mathbf{z} + \mathbf{x}_0) \leq 0, \tag{19}$$

  where $\mathbf{F}$ and $\mathbf{x}_0$ are such that: $\mathbf{A}\mathbf{x} = \mathbf{b} \Longleftrightarrow \mathbf{x} = \mathbf{F}\mathbf{z} + \mathbf{x}_0$ for some $\mathbf{z}$

Equivalent Problems

- *introducing equality constraints*

$$\min_{\mathbf{x}} \; f_0(\mathbf{A}_0\mathbf{x} + \mathbf{b}_0) \quad \text{s.t. } f_i(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i) \le 0, \; i = 1..m \tag{20}$$

is equivalent to

$$\min_{\mathbf{x},\mathbf{y}_i} \; f_0(\mathbf{y}_0) \quad \text{s.t. } f_i(\mathbf{y}_i) \le 0, \; i = 1..m \tag{21}$$

$$\mathbf{y}_i = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \; i = 0..m$$

Equivalent Problems

- *introducing slack variables for linear inequalities*

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{s.t.} & \mathbf{a}_i^T \mathbf{x} \leq \mathbf{b}_i, \quad i = 1, \ldots, m \end{array} \tag{22}$$

is equivalent to

$$\begin{array}{ll} \min \text{ (over } x, s) & f_0(\mathbf{x}) \\ \text{s.t.} & \mathbf{a}_i^T \mathbf{x} + s_i = b_i, \quad i = 1, \ldots, m \\ & s_i \geq 0, \quad i = 1, \ldots m \end{array} \tag{23}$$

Equivalent Problems

- **epigraph form:** standard form convex problem is equivalent to

$$\begin{aligned} \min \text{ (over } \mathbf{x}, t) \quad & t \\ \text{s.t.} \quad & f_0(\mathbf{x}) - t \leq 0 \\ & f_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{24}$$

Equivalent Problems

- **epigraph form:** standard form convex problem is equivalent to

$$\begin{array}{ll} \min \text{ (over } \mathbf{x}, t) & t \\ \text{s.t.} & f_0(\mathbf{x}) - t \leq 0 \\ & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{array} \tag{24}$$

- **minimizing over some variables**

$$\begin{array}{ll} \min & f_0(\mathbf{x}_1, \mathbf{x}_2) \\ \text{s.t.} & f_i(\mathbf{x}_1) \leq 0, \quad i = 1, \ldots, m \end{array} \tag{25}$$

is equivalent to

$$\begin{array}{ll} \min & \tilde{f}_0(\mathbf{x}1) \\ \text{s.t.} & f_i(\mathbf{x}_1) \leq 0, \quad i = 1, \ldots, m \end{array} \tag{26}$$

where $\tilde{f}_0(\mathbf{x}_1) = \min_{\mathbf{x}_2} f_0(\mathbf{x}_1, \mathbf{x}_2)$

Quasiconvex optimization

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \le 0, \quad i = 1, \dots, m \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{27}$$

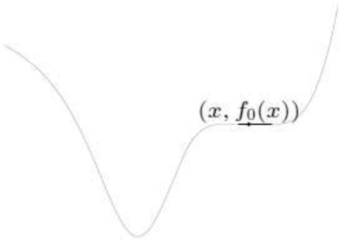
with f_0 quasiconvex, $f_1, \dots, f_m$ convex

Quasiconvex optimization

$$
\begin{aligned}
\min \quad & f_0(\mathbf{x}) \\
\text{s.t.} \quad & f_i(\mathbf{x}) \le 0, \quad i = 1, \ldots, m \\
& \mathbf{A}\mathbf{x} = \mathbf{b}
\end{aligned} \tag{27}
$$

with f_0 quasiconvex, $f_1, \ldots, f_m$ convex

Can have locally optimal points that are not (globally) optimal



Convex representation of sublevel sets of f_0

If f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

- $\phi_t(\mathbf{x})$ is convex in $\mathbf{x}$ for fixed t
- t -sublevel set of f_0 is 0 -sublevel set of ϕ_t, i.e.,

$$f_0(\mathbf{x}) \leq t \quad \Longleftrightarrow \quad \phi_t(\mathbf{x}) \leq 0 \tag{28}$$

Convex representation of sublevel sets of f_0

If f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

- $\phi_t(\mathbf{x})$ is convex in $\mathbf{x}$ for fixed t
- t -sublevel set of f_0 is 0 -sublevel set of ϕ_t, i.e.,

$$f_0(\mathbf{x}) \leq t \quad \Longleftrightarrow \quad \phi_t(\mathbf{x}) \leq 0 \tag{28}$$

Example:

$$f_0(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})} \tag{29}$$

with p convex, q concave, and $p(x) \geq 0, q(x) > 0$ on dom f_0
can take $\phi_t(\mathbf{x}) = p(\mathbf{x}) - tq(\mathbf{x})$:

- for $t \geq 0, \phi_t$ convex in $\mathbf{x}$
- $p(\mathbf{x})/q(\mathbf{x}) \leq t \leftrightarrow \phi_t(\mathbf{x}) \leq 0$

Quasiconvex optimization via convex feasibility

$$(\mathrm{P}) \quad \text{find } \mathbf{x}: \ \phi_t(\mathbf{x}) \leq 0, \quad f_i(\mathbf{x}) \leq 0, \quad \mathbf{Ax} = \mathbf{b} \tag{30}$$

- for fixed t, a convex feasibility problem in $\mathbf{x}$
- if feasible, $p^\star \leq t$; if infeasible, $p^\star \geq t$
- Why ?

Bisection method

$$\text{(P)} \quad \text{find } \mathbf{x}: \; \phi_t(\mathbf{x}) \leq 0, \quad f_i(\mathbf{x}) \leq 0, \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

Bisection method for quasiconvex optimization

given $l \leq p^\star, u \geq p^\star,$ tolerance $\epsilon > 0$
repeat

1. $t := (l + u)/2$
2. solve the convex feasibility problem (P).
3. **if** (P) is feasible, $u := t$; **else** $l := t$.

until $u - l \leq \epsilon$

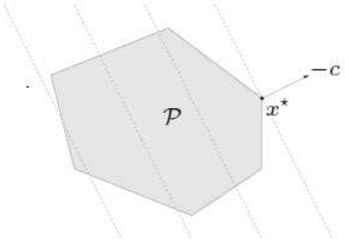
---

requires exactly $\left\lceil \log_2 \frac{u-l}{\epsilon} \right\rceil$ iterations (u, l are initial bounds)

Linear program (or problem, LP)

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} + d \\ \text{s.t.} \quad & \mathbf{G}\mathbf{x} \preceq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{31}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron (why?)



Examples

Diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \succeq \mathbf{b}, \quad \mathbf{x} \succeq 0 \end{aligned} \tag{32}$$

Examples

Diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \succeq \mathbf{b}, \quad \mathbf{x} \succeq 0 \end{array} \tag{32}$$

Piecewise-linear minimization

$$\min \max_{i=1,\ldots,m} \left(\mathbf{a}_i^T\mathbf{x} + b_i\right) \tag{33}$$

equivalent to an LP

$$\begin{array}{ll} \min & t \\ \text{s.t.} & \mathbf{a}_i^T\mathbf{x} + b_i \leq t, \quad i = 1, \ldots, m \end{array} \tag{34}$$

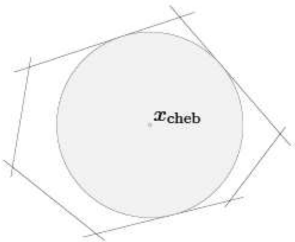
Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \left\{ \mathbf{x} : \mathbf{a}_i^T \mathbf{x} \le b_i, i = 1, \dots, m \right\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{ \mathbf{x}_c + \mathbf{u} : \|\mathbf{u}\|_2 \le r \}$$



- $\mathbf{a}_i^T \mathbf{x} \le b_i$ for all $\mathbf{x} \in \mathcal{B}$ iff

$$\max \left\{ \mathbf{a}_i^T (\mathbf{x}_c + \mathbf{u}) : \|\mathbf{u}\|_2 \le r \right\} = \mathbf{a}_i^T \mathbf{x}_c + r \|\mathbf{a}_i\|_2 \le b_i \tag{35}$$

- hence, $\mathbf{x}_c, r$ can be determined by solving the *LP*

$$\begin{array}{ll} \max & r \\ \text{s.t.} & \mathbf{a}_i^T \mathbf{x}_c + r \|\mathbf{a}_i\|_2 \le b_i, \quad i = 1, \dots, m \end{array} \tag{36}$$

Linear-fractional problem

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{s.t.} & \mathbf{Gx} \preceq \mathbf{h},\ \ \mathbf{Ax} = \mathbf{b} \end{array} \tag{37}$$

linear-fractional problem

$$f_0(\mathbf{x}) = \frac{\mathbf{c}^T\mathbf{x} + d}{\mathbf{e}^T\mathbf{x} + f}, \quad \text{dom}\ \ f_0(\mathbf{x}) = \left\{\mathbf{x} : \mathbf{e}^T\mathbf{x} + f > 0\right\} \tag{38}$$

- a quasiconvex optimization problem; can be solved by bisection

Linear-fractional problem

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{s.t.} & \mathbf{Gx} \preceq \mathbf{h},\ \mathbf{Ax} = \mathbf{b} \end{array} \tag{37}$$

linear-fractional problem

$$f_0(\mathbf{x}) = \frac{\mathbf{c}^T\mathbf{x} + d}{\mathbf{e}^T\mathbf{x} + f}, \quad \text{dom } f_0(\mathbf{x}) = \left\{\mathbf{x} : \mathbf{e}^T\mathbf{x} + f > 0\right\} \tag{38}$$

- a quasiconvex optimization problem; can be solved by bisection
- equivalent to the following LP (in $\mathbf{y}, z$)

$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{y} + dz \\ \text{s.t.} & \mathbf{Gy} \preceq hz,\ Ay = bz,\ \mathbf{e}^T\mathbf{y} + fz = 1,\ z \geq 0 \end{array} \tag{39}$$

- i.e. **non-convex** $\Rightarrow$ **convex** P !

Generalized linear-fractional program

$$f_0(\mathbf{x}) = \max_{i=1,\ldots,r} \frac{\mathbf{c}_i^T\mathbf{x} + d_i}{\mathbf{e}_i^T\mathbf{x} + f_i}, \quad \text{dom } f_0(\mathbf{x}) = \left\{\mathbf{x} : \mathbf{e}_i^T\mathbf{x} + f_i > 0, i = 1, \ldots, r\right\} \tag{40}$$

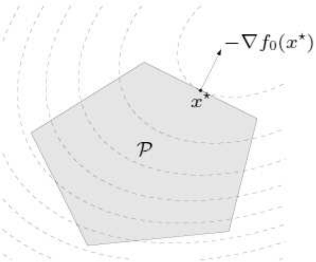
a quasiconvex optimization problem; can be solved by bisection

example: Von Neumann model of a growing economy

Quadratic program (QP)

$$\begin{array}{ll} \min & \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{q}^T\mathbf{x} + r \\ \text{s.t.} & \mathbf{G}\mathbf{x} \preceq \mathbf{h},\ \mathbf{A}\mathbf{x} = \mathbf{b} \end{array} \quad (41)$$

- $\mathbf{P} \geq 0$, so the objective is convex quadratic (what is not?)
- minimize a convex quadratic function over a polyhedron



Example: least-squares

$$\min \quad \|\mathbf{Ax} - \mathbf{b}\|_2^2 \tag{42}$$

- analytical solution: $\mathbf{x}^\star = \mathbf{A}^\dagger \mathbf{b}$, $\mathbf{A}^\dagger$ is pseudo-inverse
- can add linear constraints, e.g., $\mathbf{x}_l \leq \mathbf{x} \leq \mathbf{x}_u$

Example: linear program with random cost

$$\begin{aligned} \min \quad & \bar{\mathbf{c}}^T x + \gamma \mathbf{x}^T \mathbf{\Sigma} \mathbf{x} = \mathbb{E}\{\mathbf{c}^T \mathbf{x}\} + \gamma \operatorname{var}\left(\mathbf{c}^T \mathbf{x}\right) \\ \text{s.t.} \quad & \mathbf{G}\mathbf{x} \preceq \mathbf{h}, \quad \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{43}$$

- $\mathbf{c}$ is random vector with mean $\bar{\mathbf{c}}$ and covariance $\mathbf{\Sigma}$
- $\mathbf{c}^T\mathbf{x}$ is random variable with mean $\bar{\mathbf{c}}^T\mathbf{x}$ and variance $\mathbf{x}^T\mathbf{\Sigma}\mathbf{x}$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Quadratically constrained quadratic program (QCQP)

$$\begin{aligned} \min \quad & \mathbf{x}^T\mathbf{P}_0\mathbf{x} + \mathbf{q}_0^T\mathbf{x} + r_0 \\ \text{s.t.} \quad & \mathbf{x}^T\mathbf{P}_i\mathbf{x} + \mathbf{q}_i^T\mathbf{x} + r_i \leq 0 \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{44}$$

- $\mathbf{P}_1..\mathbf{P}_m \geq 0$, objective and constraints are convex quadratic
- if $\mathbf{P}_1..\mathbf{P}_m > 0$, feasible set is intersection of ellipsoids and an affine set (if not?)

Second-order cone programming

$$\begin{aligned} \min \quad & \mathbf{f}^T \mathbf{x} \\ \text{s.t.} \quad & \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^T \mathbf{x} + d_i, \quad i = 1, \dots, m \\ & \mathbf{F}\mathbf{x} = \mathbf{g} \end{aligned} \tag{45}$$

- inequalities are called second-order cone (SOC) constraints:

$$\left(\mathbf{A}_i \mathbf{x} + \mathbf{b}_i, \mathbf{c}_i^T \mathbf{x} + \mathbf{d}_i\right) \in \text{second-order cone}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP
- more general than QCQP and LP

Robust linear programming

The parameters in optimization problems are often uncertain, e.g.

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i^T \mathbf{x} \leq \mathbf{b}_i, \quad i = 1, \ldots, m \end{aligned} \tag{46}$$

with uncertainty in $\mathbf{c}$, $\mathbf{a}_i$, b_i

Robust linear programming

The parameters in optimization problems are often uncertain, e.g.

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i^T \mathbf{x} \le \mathbf{b}_i, \quad i = 1, \dots, m \end{aligned} \tag{46}$$

with uncertainty in $\mathbf{c}$, $\mathbf{a}_i$, b_i

Two common approaches to handling uncertainty (in $\mathbf{a}_i$, for simplicity)

- deterministic model: constraints must hold for all $\mathbf{a}_i \in \mathcal{E}_i$

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i^T \mathbf{x} \le \mathbf{b}_i, \ \forall\, \mathbf{a}_i \in \mathcal{E}_i, \quad i = 1, \dots, m \end{aligned} \tag{47}$$

Robust linear programming

The parameters in optimization problems are often uncertain, e.g.

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i^T\mathbf{x} \leq \mathbf{b}_i, \quad i = 1, \ldots, m \end{aligned} \tag{46}$$

with uncertainty in $\mathbf{c}$, $\mathbf{a}_i$, b_i

Two common approaches to handling uncertainty (in $\mathbf{a}_i$, for simplicity)

- deterministic model: constraints must hold for all $\mathbf{a}_i \in \mathcal{E}_i$

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i^T\mathbf{x} \leq \mathbf{b}_i, \ \forall\ \mathbf{a}_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{aligned} \tag{47}$$

- stochastic model: $\mathbf{a}_i$ is random variable; constraints must hold with probability at least η

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \Pr\left(\mathbf{a}_i^T\mathbf{x} \leq b_i\right) \geq \eta, \quad i = 1, \ldots, m \end{aligned} \tag{48}$$

Geometric programming

- **monomial function**

$$f(\mathbf{x}) = c \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad x_1, .., x_n > 0 \tag{49}$$

with $c > 0$; a_i = any real number

Geometric programming

- **monomial function**

$$f(\mathbf{x}) = c \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad x_1, .., x_n > 0 \tag{49}$$

  with $c > 0$; $a_i =$ any real number

- **posynomial function:** sum of monomials

$$f(\mathbf{x}) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}} \tag{50}$$

Geometric programming

- **monomial function**

$$f(\mathbf{x}) = c \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad x_1, .., x_n > 0 \tag{49}$$

  with $c > 0$; a_i = any real number

- **posynomial function:** sum of monomials

$$f(\mathbf{x}) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}} \tag{50}$$

- **geometric program (GP)**

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \le 1, \ h_i(\mathbf{x}) = 1, \quad i = 1, \ldots, p \end{aligned} \tag{51}$$

  with f_i posynomial, h_i monomial

- Q: convex or not?

Geometric program in convex form

Change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(\mathbf{x}) = c \cdot x_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f\left(e^{y_1}, \ldots, e^{y_n}\right) = \mathbf{a}^T \mathbf{y} + b \quad (b = \log c) \tag{52}$$

Geometric program in convex form

Change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(\mathbf{x}) = c \cdot x_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f\left(e^{y_1}, \dots, e^{y_n}\right) = \mathbf{a}^T \mathbf{y} + b \quad (b = \log c) \tag{52}$$

- posynomial $f(\mathbf{x}) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$ transforms to

$$\log f\left(e^{y_1}, \dots, e^{y_n}\right) = \log\left(\sum_{k=1}^{K} e^{\mathbf{a}_k^T \mathbf{y} + b_k}\right), \quad b_k = \log c_k \tag{53}$$

Geometric program in convex form

Change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(\mathbf{x}) = c \cdot x_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f\left(e^{y_1}, \dots, e^{y_n}\right) = \mathbf{a}^T \mathbf{y} + b \quad (b = \log c) \tag{52}$$

- posynomial $f(\mathbf{x}) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$ transforms to

$$\log f\left(e^{y_1}, \dots, e^{y_n}\right) = \log \left(\sum_{k=1}^{K} e^{\mathbf{a}_k^T \mathbf{y} + b_k} \right), \quad b_k = \log c_k \tag{53}$$

- now, **convex or not?**

Geometric program in convex form

- geometric program transforms to convex problem

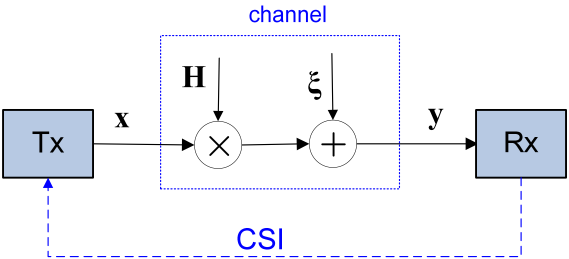
$$
\begin{aligned}
\min \quad & \log\left(\sum_{k=1}^{K} \exp\left(\mathbf{a}_{0k}^T \mathbf{y} + b_{0k}\right)\right) \\
\text{s.t.} \quad & \log\left(\sum_{k=1}^{K} \exp\left(\mathbf{a}_{ik}^T \mathbf{y} + b_{ik}\right)\right) \leq 0, \quad i = 1, \ldots, m \\
& \mathbf{G}\mathbf{y} + \mathbf{d} = 0
\end{aligned} \tag{54}
$$

Multi-antenna Gaussian channel

Example: maximizing rate (MI) in multi-antenna Gaussian channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \boldsymbol{\xi} \tag{55}$$

$\mathbf{x}, \mathbf{y}$ = input (Tx) and output (Rx), $\boldsymbol{\xi}$ = noise, $\mathbf{H}$ = channel matrix



Semidefinite problem (SDP)

Example: maximizing rate (MI) in multi-antenna Gaussian channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \boldsymbol{\xi}, \ \text{MI} = \log|\mathbf{I} + \mathbf{W}\mathbf{R}| \tag{56}$$

$\mathbf{x}, \mathbf{y}$ = input (Tx) and output (Rx), $\boldsymbol{\xi}$ = noise, $\mathbf{H}$ = channel matrix

$$\max_{\mathbf{R}} \ \text{MI} = \log|\mathbf{I} + \mathbf{W}\mathbf{R}| \text{ s.t. } \mathbf{R} \geq 0, tr\mathbf{R} \leq P_T \tag{57}$$

$\mathbf{R}$ = Tx (input) covariance matrix, $tr\mathbf{R}$ = its power
P_T = max. Tx power
$\mathbf{W} = \mathbf{H}^+\mathbf{H}$ = channel Gram matrix

- Very important in wireless communications (WiFi, 5G)
- Q: convex or not?

Maximizing rate in multi-antenna Gaussian channel

Can add extra constrains:

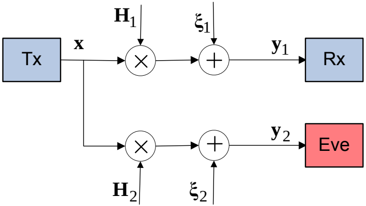
$$\max_{\mathbf{R}} \text{MI} = \log |\mathbf{I} + \mathbf{W}\mathbf{R}| \quad (58)$$
$$\text{s.t. } \mathbf{R} \geq 0, tr\mathbf{R} \leq P_T, r_{ii} \leq P_1, tr(\mathbf{W}_2\mathbf{R}) \leq P_I$$

$r_{ii} \leq P_1$ - per antenna power constraint
$tr(\mathbf{W}_2\mathbf{R}) \leq P_I$ - interference power constraint

Maximizing secrecy rate

- wire-tap MIMO channel model

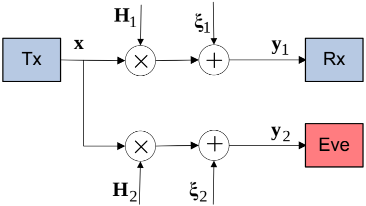
$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{x} + \xi_1, \quad \mathbf{y}_2 = \mathbf{H}_2\mathbf{x} + \xi_2 \tag{59}$$



Maximizing secrecy rate

- wire-tap MIMO channel model

$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{x} + \xi_1, \quad \mathbf{y}_2 = \mathbf{H}_2\mathbf{x} + \xi_2 \tag{59}$$



- secrecy rate maximization

$$\max_{\mathbf{R}} \log \frac{|\mathbf{I} + \mathbf{W}_1\mathbf{R}|}{|\mathbf{I} + \mathbf{W}_2\mathbf{R}|} \text{ s.t. } \mathbf{R} \geq 0, tr\mathbf{R} \leq P_T \tag{60}$$

- convex or not?

More Examples (see Boyd & Vandenberghe)

- max. eigenvalue minimization
- matrix norm minimization
- general vector optimization problem
- multicriterion optimization (optimal and Pareto-optimal points, scalarization)