Assignment #4

Due: by 4pm, Apr. 8 (in-class). Late or email submissions will not be accepted.

Reading: Chapter 5 of the course textbook (S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004). Study carefully all examples, make sure you understand them and can repeat them with the book closed. You are encouraged to at least read all end-of-chapter problems and attempt to solve more than actually asked below. Remember the learning efficiency pyramid!

1) Consider the following problem with variable $x \in \mathbf{R}$

$$\min\ x^2 + 1\ \text{ s.t. } (x-2)(x-3) \le 0$$

a) Is it a convex problem?

b) Find the feasible set, the optimal value, and the optimal point.

c) Plot the objective $x^2 + 1$ versus x . On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \ge \min_x L(x, \lambda)$ for $\lambda \ge 0$). Derive and sketch the Lagrange dual function g.

d) State the dual problem, prove that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

e) Let $p^*(u)$ be the optimal value of the perturbed problem

$$\min\ x^2 + 1\ \text{ s.t. } (x-2)(x-3) \le u,$$

as a function of the perturbation parameter u. Plot $p^*(u)$. Verify that $dp^*(0)/du = -\lambda^*$.

2) Consider the following problem with variable $x \in \mathbf{R}^2$

$$\text{min. } x_1^2 + x_2^2\ \text{ s.t. } (x_1 - 1)^2 + (x_2 - 1)^2 \le 1 \text{ , } (x_1 - 1)^2 + (x_2 + 1)^2 \le 1$$

a) Is it a convex problem?
b) Sketch the feasible set and level sets of the objective. Find the optimal point x^* and optimal value p^*.
c) Give the KKT conditions. Do there exist Lagrange multipliers λ_1^* and λ_2^* that prove that $x^*$is optimal?
d) Derive and solve the Lagrange dual problem. Does strong duality hold?
e) Does Slater's condition hold?

3) In the class, we derived the WF algorithm using the KKT conditions under the equality total power constraint, i.e. $\sum_i p_i = P_T$. Give a derivation under the inequality constraint, $\sum_i p_i \le P_T$. Prove (from the KKT conditions) that it is always optimal to use the full available power, i.e. that the inequality always holds with equality at optimal point.

4) Consider a multi-stream transmission system with 3 parallel channels and independent AWGN noise on each channel, as discussed in the class. The noise powers are $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 1$.

a. Sketch a block diagram of this system, clearly indicate signals at its key points and explain what they are. Write down the equations for the rate on each channel and the total rate, for given Tx signal powers p_i , $i = 1,2,3$.

b. State the optimization problem to maximize the total (sum) rate subject to the total power constraint ($\le P_T$) . Is it a convex problem?

c. Write down the water-filling solution to this problem; explain how the dual variable is determined.

d. Implement the WF algorithm in a software code (of your choice, Matlab is a good choice). Plot the optimal power allocation $p_i^*(P_T)$, $i=1,2,3$, vs. P_T (on a single graph), and, on a separate graph, the total rate $R(P_T)$ vs. P_T, for $0 \le P_T \le 100$. Explain what you see. Hint: a bisection algorithm can be used to efficiently find the dual variable for a given P_T.

5) Consider a convex problem with no equality constraints,

$$\min f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0, \;\; i = 1, \dots, m$$

Assume that $x^* \in \mathbf{R}^n$ and $\lambda^* \in \mathbf{R}^m$ satisfy the KKT conditions

$$f_i(x^*) \le 0, \quad i = 1, \dots, m$$

$$\lambda_i^* \ge 0, \quad i = 1, \dots, m$$

$$\lambda_i^* f_i(x^*) = 0, \quad i = 1, \dots, m$$

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) = 0.$$

Show that

$$\nabla f_0(x^*)^T (x - x^*) \ge 0$$

for all feasible x. In other words the KKT conditions imply the simple optimality criterion of Lecture 4.

6) Let $f_0, f_1, \dots, f_m$ be convex functions. Show that the function

$$p^*(u, v) = \min\{f_0(x): \text{ for } x \text{ satisfying } f_i(x) \le u_i, i = 1, \dots, m, Ax - b = v\}$$

is convex. This function is the optimal value of the perturbed problem as a function of the perturbations u and v.

<u>Important rules (deviation will be penalized):</u>

<u>Please give your solutions in the order indicated above. Start each new problem on a new page (no 2 problems on the same page).</u>

Please include in your solutions all the intermediate results and their numerical values (if applicable). Detailed solutions are required, not just the final answers.

Make sure your handwriting is readable and is sufficiently large so it can be read without a microscope; otherwise, it will be ignored.

<u>Plagiarism</u> (i.e. "cut-and-paste" from a student to a student, other forms of "borrowing" the material for the assignment) is absolutely unacceptable and will be penalized. Each student is expected to submit his own solutions. If two (or more) identical or almost identical sets of solutions are found, each student involved receives 0 (zero) for that particular assignment. If this happens twice, the students involved receive 0 (zero) for the entire assignment component of the course in the marking scheme and the case will be send to the Dean's office for further investigation.