# Assignment #3

**Due:** by 4pm, Mar. 25 (in-class). Late or email submissions will not be accepted.

**Reading**: Chapter 4 of the course textbook (S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004). Study carefully all examples, make sure you understand them and can repeat them with the book closed. You are encouraged to at least read all end-of-chapter problems and attempt to solve more than actually asked below. Remember the learning efficiency pyramid!

1) Consider the following problem

$$\min\ f_0(x_1, x_2) \quad \text{s.t.} \quad 2x_1 + x_2 \geq 1,\ x_1 + 3x_2 \geq 1, \quad x_1 \geq 0, x_2 \geq 0.$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value. Additionally, indicate supporting hyperplane in each case.

(a) $f_0(x_1, x_2) = 2x_1 + x_2$.
(b) $f_0(x_1, x_2) = -x_1 - 2x_2$.
(c) $f_0(x_1, x_2) = x_1$.
(d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.
(e) $f_0(x_1, x_2) = 9{x_1}^2 + {x_2}^2$.

*2)* A convex optimization problem can have only linear equality constraint functions. In some special cases, however, it is possible to handle convex equality constraint functions, *i.e.*, constraints of the form $h(x) = 0$, where $h$ is convex. We explore this idea in the following problem.

$$\text{(P1)} \qquad \min\ f_0(x) \quad \text{s.t.}\ f_i(x) \leq 0,\ \ h(x) = 0,\ \ i = 1, \ldots, m$$

where $f_i$ and $h$ are convex functions.

(a) Is (P1) a convex problem?

Now, consider a related problem

$$\text{(P2)} \qquad \min\ f_0(x) \quad \text{s.t.}\ f_i(x) \leq 0,\ \ h(x) \leq\ 0,\ \ i = 1, \ldots, m$$

where the equality constraint has been relaxed to a convex inequality.

(b) Is this problem convex?

Now assume that for any optimal solution $x^*$ of (P2), we have $h(x^*) = 0$, *i.e.*, the inequality $h(x) \leq 0$ is always active at the solution. Then,

(c) prove that any solution of (P2) also solves (P1).

(d) prove that the above assumption holds if there is an index $r$ such that

- $f_0$ is monotonically increasing in $x_r$
- $f_1, \ldots, f_m$ are nondecreasing in $x_r$
- $h$ is monotonically decreasing in $x_r$.

3) Give an explicit solution of each of the following LPs.

(a) *Minimizing a linear function over a rectangle:* $\min\ c^T x \quad \text{s.t.}\ \ u \leq x \leq l$, where $l$ and $u$ satisfy $u \leq l$.

(b) *Minimizing a linear function over a halfspace:* $\min\ c^T x \quad \text{s.t.}\ \ a^T x \leq b$, where $a \neq 0$.

*(c) Minimizing a linear function over an affine set:* $\min\ c^T x \quad \text{s.t.}\ \ Ax = b$.

(d) *Minimizing a linear function over the probability simplex:* min $c^T x$ s.t. $1^T x = 1,\ x \geq 0$. What happens if the equality constraint is replaced by an inequality $1^T x \leq 1$?

4) *Problems involving $\ell_1$- and $\ell_\infty$-norms*. Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP.

(a) Minimize $\|Ax - b\|_\infty$ ($\ell_\infty$-norm approximation).

(b) Minimize $\|Ax - b\|_1$ ($\ell_1$-norm approximation).

5) Give an explicit solution of each of the following QCQPs.

*(a) Minimizing a linear function over an ellipsoid centered at the origin.*

$$\min\ c^T x \qquad \text{s.t. } x^T A x \leq 1$$

where $A \succ 0$ and $c \neq 0$. What is the solution if the problem is not convex ($A \notin \mathbf{S}_+^n$)?

*(b) Minimizing a linear function over an ellipsoid.*

$$\min\ c^T x \ \text{ s.t. } (x - x_c)^T A (x - x_c) \leq 1$$

where $A \succ 0$ and $c \neq 0$.

*(c) Minimizing a quadratic form over an ellipsoid centered at the origin.*

$$\min\ \ x^T B x \qquad \text{s.t. } x^T A x \leq 1$$

where $A \succ 0$ and $B \succeq 0$. Also consider the nonconvex extension with $B \notin \mathbf{S}_+^n$. (See Sec B.1 of the textbook.)

6) Express the following problems as convex optimizations problems.

(a) Minimize $\max\{2p(x), 3q(x)\}$, where $p$ and $q$ are posynomials.
(b) Minimize $3^{p(x)} + 2^{q(x)}$, where $p$ and $q$ are posynomials.

7) *Capacity of a communication channel*. We consider a communication channel, with random input $X(t) \in \{1, \dots, n\}$, and random output $Y(t) \in \{1, \dots, m\}$, where $t = 1,2, \dots,$ is discrete time. The relation between the input and the output is given statically by conditional probabilities:

$$p_{ij} = \Pr(Y(t) = i | X(t) = j),\ \ i = 1, \dots, m,\ \ j = 1, \dots, n.$$

The matrix $P \in \mathbf{R}^{m \times n}$ is called the *channel probability transition matrix*, and the channel is called a *discrete memoryless channel*.

The celebrated result of Shannon states that information can be sent over the communication channel, with arbitrary small probability of error, at any rate less than that a number $C$, called the *channel capacity*. Shannon also showed that the capacity of a discrete memoryless channel can be found by solving an optimization problem. Assume that $X$ has a probability distribution denoted $x \in \mathbf{R}^n$, *i.e.*,

$$x_j = \Pr(X = j), \quad j = 1, \dots, n.$$

The *mutual information* between $X$ and $Y$ is given by

$$I(X;Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j p_{ij} \log_2 \frac{p_{ij}}{\sum_{k=1}^{n} x_k p_{ik}}$$

Then the channel capacity $C$ is given by

$$C = \max_X I(X;Y)$$

where the supremum is over all possible probability distributions for the input $X$, *i.e.*, over $x \geq 0, 1^T x = 1$.

Show how the channel capacity can be computed using convex optimization.

*Hint*. Introduce the variable $y = Px$, which gives the probability distribution of the output Y, and show that the mutual information can be expressed as

$$I(X;Y) = c^T x - \sum_{i=1}^{m} y_i \log_2 y_i,$$

where $c_j = \sum_{i=1}^{m} p_{ij} \log_2 p_{ij}$ ,  $j = 1, \dots, n$.

**Important rules (deviation will be penalized):**

*Please give your solutions in the order indicated above. Start each new problem on a new page (no 2 problems on the same page). Staple the sheets.*

Please include in your solutions all the intermediate results and their numerical values (if applicable). Detailed solutions are required, not just the final answers.

Make sure your handwriting is readable and is sufficiently large so it can be read without a microscope; otherwise, it will be ignored.

**Plagiarism** (i.e. "cut-and-paste" from a student to a student, other forms of "borrowing" the material for the assignment) is absolutely unacceptable and will be penalized. Each student is expected to submit his own solutions. If two (or more) identical or almost identical sets of solutions are found, each student involved receives 0 (zero) for that particular assignment. If this happens twice, the students involved receive 0 (zero) for the entire assignment component of the course in the marking scheme and the case will be send to the Dean's office for further investigation.