Assignment #2

Due: by 4pm, Feb. 26 (in-class). Late or email submissions will not be accepted.

Reading: Chapter 3 of the course textbook (S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004). Study carefully all examples, make sure you understand them and can repeat them with the book closed. You are encouraged to at least read all end-of-chapter problems and attempt to solve more than actually asked below. Remember the learning efficiency pyramid!

1) If $f(x)$ is a convex function of scalar argument x, and $a, b \in \textbf{dom} f$ with $a < b$.

   (a) Show that for all $x \in [a, b]$, the following holds

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

   (b) Further show that

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

   and draw a sketch that illustrates this inequality.

   (c) If $f(x)$ is differentiable, use the inequalities in (b) to show that

$$f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b).$$

   (d) If $f(x)$ is twice differentiable, use (c) to show that $f''(a) \geq 0$ and $f''(b) \geq 0$.

2) Level sets of functions f and g are shown in Fig. 2(a) and 2(b) below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc. Could f be convex, concave, quasiconvex, quasiconcave ? What about g ? Explain your answer for each case.

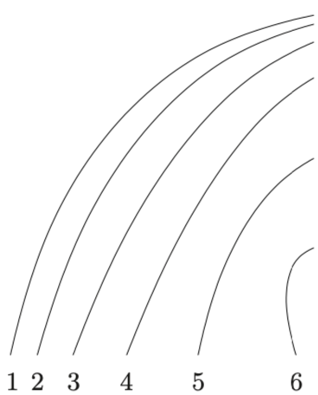

Fig. 2(a): level sets of function f

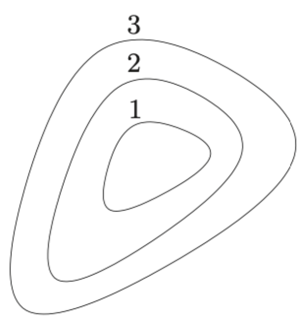

Fig. 2(b): level sets of function g

3) For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

   (a) $f(x) = e^x - 1$

   (b) $f(x_1, x_2) = x_1 x_2$ for $x_1, x_2 \geq 0$.

(c) $f(x_1, x_2) = (x_1 x_2)^{-1}$ on for $x_1, x_2 > 0$.

4) Using the composition rules, show that the following function is convex.

$f(x, u, v) = -\sqrt{uv - x^T x}$ on $\mathbf{dom}\, f = \{(x, u, v) |\, uv > x^T x, u, v > 0\}$. Use the fact that $x^T x / u$ is convex in (x, u) for $u > 0$, and that $-\sqrt{x_1 x_2}$ is convex for $x_1, x_2 > 0$.

5) Some functions on probability simplex. Let x be a real-valued random variable which takes values in $\{a_1, \dots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\Pr(x = a_i) = p_i,\ i = 1, \dots, n$. For each of the following functions of $\boldsymbol{p}$ (on the probability simplex $\sum_{i=1}^{n} p_i = 1, p_i \geq 0$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

(a) $f(\mathbf{p}) = \mathbf{E}\{x\}$ (mean value of x as a function of $\mathbf{p}$).

(b) $f(\mathbf{p}) = \Pr(\alpha \leq x \leq \beta)$ (probability that x falls within interval $[\alpha, \beta]$).

(c) $f(\mathbf{p}) = \mathrm{var}\{x\} = \mathbf{E}\{x - \mathbf{E}\{x\}\}^2$. (variance of x)

6) Use the first and second-order conditions for quasiconvexity given in section 3.4.3 of the course textbook to prove quasiconvexity of the function $f(x) = -x_1 x_2$ for $x_1, x_2 > 0$.

7) Show that the following functions are log-concave.

$$(a)\ f(x) = \frac{e^x}{1 + e^x}, \qquad (b)\ f(\boldsymbol{x}) = \frac{\prod_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i} \text{ for } x_i > 0$$

Important rules (deviation will be penalized):

Please give your solutions in the order indicated above. Start each new problem on a new page (no 2 problems on the same page).

Please include in your solutions all the intermediate results and their numerical values (if applicable). Detailed solutions are required, not just the final answers.

Make sure your handwriting is readable and is sufficiently large so it can be read without a microscope; otherwise, it will be ignored.

Plagiarism (i.e. “cut-and-paste” from a student to a student, other forms of “borrowing” the material for the assignment) is absolutely unacceptable and will be penalized. Each student is expected to submit his own solutions. If two (or more) identical or almost identical sets of solutions are found, each student involved receives 0 (zero) for that particular assignment. If this happens twice, the students involved receive 0 (zero) for the entire assignment component of the course in the marking scheme and the case will be send to the Dean’s office for further investigation.