Max. SNR/MMSE V-BLAST

Consider regular (ZF) V-BLAST first.
Basic channel model:

$$\mathbf{y} = \mathbf{Hx} + \boldsymbol{\xi} = \sum_{i=1}^{m} \mathbf{h}_i x_i + \boldsymbol{\xi} \quad (1)$$

Detection step i: assume the T_x symbols $[x_1 ... x_{i-1}]$ has been correctly detected,

$$\widehat{x_j} = x_j, \quad j = 1...i-1 \quad (2)$$

Subtract the contribution of already detected symbols from $\mathbf{y}$,

$$\mathbf{y}' = \mathbf{y} - \sum_{j=1}^{i-1} x_j \mathbf{h}_j = \sum_{k=i}^{m} \mathbf{h}_k x_k + \boldsymbol{\xi} = \mathbf{H}_{(i-1)} \mathbf{x}_{(i-1)} + \boldsymbol{\xi} \quad (3)$$

where $\mathbf{H}_{i-1} = [\mathbf{h}_i, \mathbf{h}_{i+1}, .. \mathbf{h}_m]$, $\mathbf{x}_{(i-1)} = [x_i, x_{i+1}, .. x_m]^T$. This is interference cancellation stage.

Next, project out interference from yet to detected symbols $\mathbf{x}_{(i)}$:

$$\mathbf{y}'' = \mathbf{P}_i \mathbf{y}' = \mathbf{P}_i \mathbf{h}_i x_i + \mathbf{P}_i \boldsymbol{\xi}, \quad (4)$$

where $\mathbf{P}_i = \mathbf{I} - \mathbf{H}_i \left(\mathbf{H}_i^+ \mathbf{H}_i\right)^{-1} \mathbf{H}_i^+$ is the projection matrix (orthogonal to span$\{\mathbf{h}_{i+1} .. \mathbf{h}_m\}$).
Finally, do MRC using $\mathbf{y}''$ to maximize output SNR:

$$\hat{y}_i = \boldsymbol{\alpha}_i^+ \mathbf{y}'' = \mathbf{h}_i^+ \mathbf{P}_i \mathbf{h}_i x_i + \mathbf{h}_i^+ \mathbf{P}_i \boldsymbol{\xi} \quad (5)$$

where $\boldsymbol{\alpha}_i = \mathbf{h}_i$ are MRC weights. (5) can be compactly expressed as:

$$\hat{y}_i = \mathbf{w}_i^+ y', \quad \mathbf{w}_i = \mathbf{P}_i \mathbf{h}_i \quad (6)$$

where we used the fact that $\mathbf{P}_i^+ = \mathbf{P}_i$.

The output SNR is:

$$\gamma_i = \frac{\left\langle \left| \mathbf{h}_i^+ \mathbf{P}_i \mathbf{h}_i \mathbf{x}_i \right|^2 \right\rangle}{\left\langle \left| \mathbf{h}_i^+ \mathbf{P}_i \boldsymbol{\xi} \right|^2 \right\rangle} = \frac{\mathbf{h}_i^+ \mathbf{P}_i \mathbf{h}_i}{\sigma_0^2} \quad (7)$$

assuming $\left\langle |x_i|^2 \right\rangle = 1$ (unit power constellation). $\hat{y}_i$ is the decision variable to find x_i.

This algorithm is sometimes called ZF (zero-forcing) V-BLAST as $\mathbf{P}_i$ cancels completely ISI (inter-stream interference) from $\{x_{i+1} ... x_m\}$. It does not minimize BER, however.

Max. SNR V-BLAST

Consider step i and find such weights $\mathbf{w}_i$ that the output SNR is maximized,

$$\hat{y}_i = \mathbf{w}_i^+ \mathbf{y}' = r_{si} + r_{\xi i}$$
$$r_{si} = \mathbf{w}_i^+ \mathbf{h}_i x_i, \quad r_{\xi i} = \sum_{k=i+1}^{m} \mathbf{w}_i^+ \mathbf{h}_k x_k + \mathbf{w}_i^+ \boldsymbol{\xi} \tag{8}$$

Output signal and noise/interference powers:

$$P_s = \left\langle \left| r_{si} \right|^2 \right\rangle = \left| \mathbf{w}_i^+ \mathbf{h}_i \right|^2$$
$$P_\xi = \left\langle \left| r_{\xi i} \right|^2 \right\rangle = \left\langle \mathbf{w}_i^+ \mathbf{H}_i \mathbf{x}\mathbf{x}^+ \mathbf{H}_i^{\;+} \mathbf{w}_i \right\rangle + \sigma_0^2 \mathbf{w}_i^+ \mathbf{w}_i = \tag{9}$$
$$= \mathbf{w}_i^+ \left(\sigma_0^2 \mathbf{I} + \mathbf{H}_i \mathbf{H}_i^{\;+} \right) \mathbf{w}_i$$

where we have used $\left\langle \mathbf{x}_{(i+1)} \mathbf{x}_{(i+1)}^+ \right\rangle = \mathbf{I}, \ \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^+ \right\rangle = \sigma_0^2 \mathbf{I}$ (i.e. i.i.d. signals and noise). Finally, the output SNR is

$$\gamma_i = \frac{P_s}{P_\xi} = \frac{\mathbf{w}_i^+ \mathbf{h}_i \mathbf{h}_i^+ \mathbf{w}_i}{\mathbf{w}_i^+ R_\xi \mathbf{w}_i} \tag{10}$$

where $\mathbf{R}_\xi = \sigma_0^2 \mathbf{I} + \mathbf{H}_i \mathbf{H}_i^{\;+}$ is noise and ISI correlation matrix.

Optimization problem:

$$\max_{\mathbf{w}_i} \gamma_i \tag{11}$$

The solution is

$$\mathbf{w}_i = \mathbf{R}_\xi^{-1} \mathbf{h}_i \tag{12}$$

and the max SNR is

$$\gamma_{i,\max} = \mathbf{h}_i^+ \mathbf{R}_\xi^{-1} \mathbf{h}_i \tag{13}$$

Compare to ZF solution (7); for large average SNR, $\sigma_0^2 \to 0$,

$$\mathbf{R}_\xi^{-1} = \left(\sigma_0^2 \mathbf{I} + \mathbf{H}_i \mathbf{H}_i^{\;+} \right)^{-1} \approx \frac{1}{\sigma_0^2} \mathbf{P}_i \tag{14}$$

and max SNR solution is very close to ZF solution (7),

$$\gamma_{i,\max} \approx \gamma_{iZF} \tag{15}$$

Max SNR solution (12)-(13) has very important property.

Theorem: Max SNR V-BLAST achieves MIMO capacity.

Proof:

$$C = \log\left|I + \frac{\rho}{m}\mathbf{H}\mathbf{H}^+\right| = \log\left|I + \frac{\rho}{m}\mathbf{H}_1\mathbf{H}_1^+ + \frac{\rho}{n}\mathbf{h}_1\mathbf{h}_1^+\right|$$

$$= \log\left|I + \frac{\rho}{m}\mathbf{H}_1\mathbf{H}_1^+\right| + \log\left|I + \frac{\rho}{m}\left(I + \frac{\rho}{n}\mathbf{H}_1\mathbf{H}_1^+\right)^{-1}\mathbf{h}_1\mathbf{h}_1^+\right| \quad (16)$$

$$= \log\left|I + \frac{\rho}{m}\mathbf{H}_1\mathbf{H}_1^+\right| + \Delta_1$$

$$\Delta_1 = \log\left(1 + \frac{\rho}{m}\mathbf{h}_1^+\left(I + \frac{\rho}{m}\mathbf{H}_1\mathbf{H}_1^+\right)^{-1}\mathbf{h}_1\right) \quad (17)$$

Note that with our normalization, $\left\langle |x_i|^2 \right\rangle = 1$,

$$\frac{\rho}{m} = \frac{1}{\sigma_0^2} \quad (18)$$

and

$$\Delta_1 = \log(1+\gamma_1), \quad \gamma_1 = \mathbf{h}_1^+\left(\sigma_0^2\mathbf{I} + \mathbf{H}_1\mathbf{H}_1^+\right)^{-1}\mathbf{h}_1 \quad (19)$$

Comparing (19) to (13), we conclude that γ_1 is the output SNR of the max SNR processing at step1 (considering T_x 2…m as sources of interference, ISI). Hence, Δ_1 is the capacity at step 1.

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Applying the same expansion to $\log\left|\mathbf{I} + \frac{\rho}{m}\mathbf{H}_1\mathbf{H}_1^+\right|$, one obtains:

$$C = \sum_{i=1}^{m}\Delta_i; \quad \begin{array}{c} \Delta_i = \log(1+\gamma_i) \\ \gamma_i = \mathbf{h}_i^+\left(\sigma_0^2\mathbf{I} + \mathbf{H}_i\mathbf{H}_i^+\right)\mathbf{h}_i \end{array} \quad (20)$$

where Δ_i is the capacity of i-th stream, and γ_i is the SNR with max SNR processing. Q.E.D.

Note: from (14), one may conclude that asymptotically, $\sigma_0^2 \ll 1$, ZF V-BLAST also achieves MIMO capacity.

MMSE BLAST

In a similar way, one may consider MMSE solution to the stream separation problem in (1)

$$\min_{\mathbf{w}_i} \varepsilon_i^2 \quad , \quad \varepsilon_i^2 = \left\langle \left|x_i - \mathbf{w}_i^+\mathbf{y}\right|^2 \right\rangle \quad (21)$$

Using

$$\frac{d\varepsilon_i^2}{d\mathbf{w}_i} = 0 \quad (22)$$

one finds MMSE weights as

$$\mathbf{w}_i = \left(\sigma_0^2\mathbf{I} + \mathbf{H}\mathbf{H}^+\right)^{-1}\mathbf{h}_i \quad (23)$$

and

$$\varepsilon_{i,\min}^2 = 1 - \mathbf{h}_i^+ \left(\sigma_0^2 \mathbf{I} + \mathbf{H}\mathbf{H}^+\right)^{-1} \mathbf{h}_i \qquad (24)$$

After some manipulations, it can be shown that max SNR and MMSE weights are related as

$$\mathbf{w}_{MMSE} = \frac{\mathbf{w}_{SNR}}{1+\gamma_i}, \quad \gamma_i = \mathbf{h}_i^+ \mathbf{R}_\xi^{-1} \mathbf{h}_i \qquad (25)$$

and, hence, MMSE solution also provides max SNR.

Important relationship between min MMSE and max SNR:

$$\frac{1}{\varepsilon_{\min,i}^2} = 1 + \gamma_i \qquad (26)$$

Exercise: prove (14), (23), (25), (26).