Synthesis of Linear Arrays

ULA pattern is not acceptable in many cases (such as high side lobes, wide main beam, etc.).

Is it possible to improve it ? Yes!

How? –> use non-uniform weighting to decrease side lobe level.

Some examples: raised cosine weighting

$$w_i = c\left[p + (1-p)\cos\frac{\pi i}{N} \right], \quad i = -\frac{N-1}{2},\ldots,\frac{N-1}{2}$$

N – assumed to be odd, p - a parameter, $0 \le p \le 1$ (5.1)

c - constant , $c = \frac{p}{N} + (1-p)\sin\frac{\pi}{2N}$, so that $F(0) = 1$

As p decreases, the height of 1st side lobe decreases and the main beam width increases.

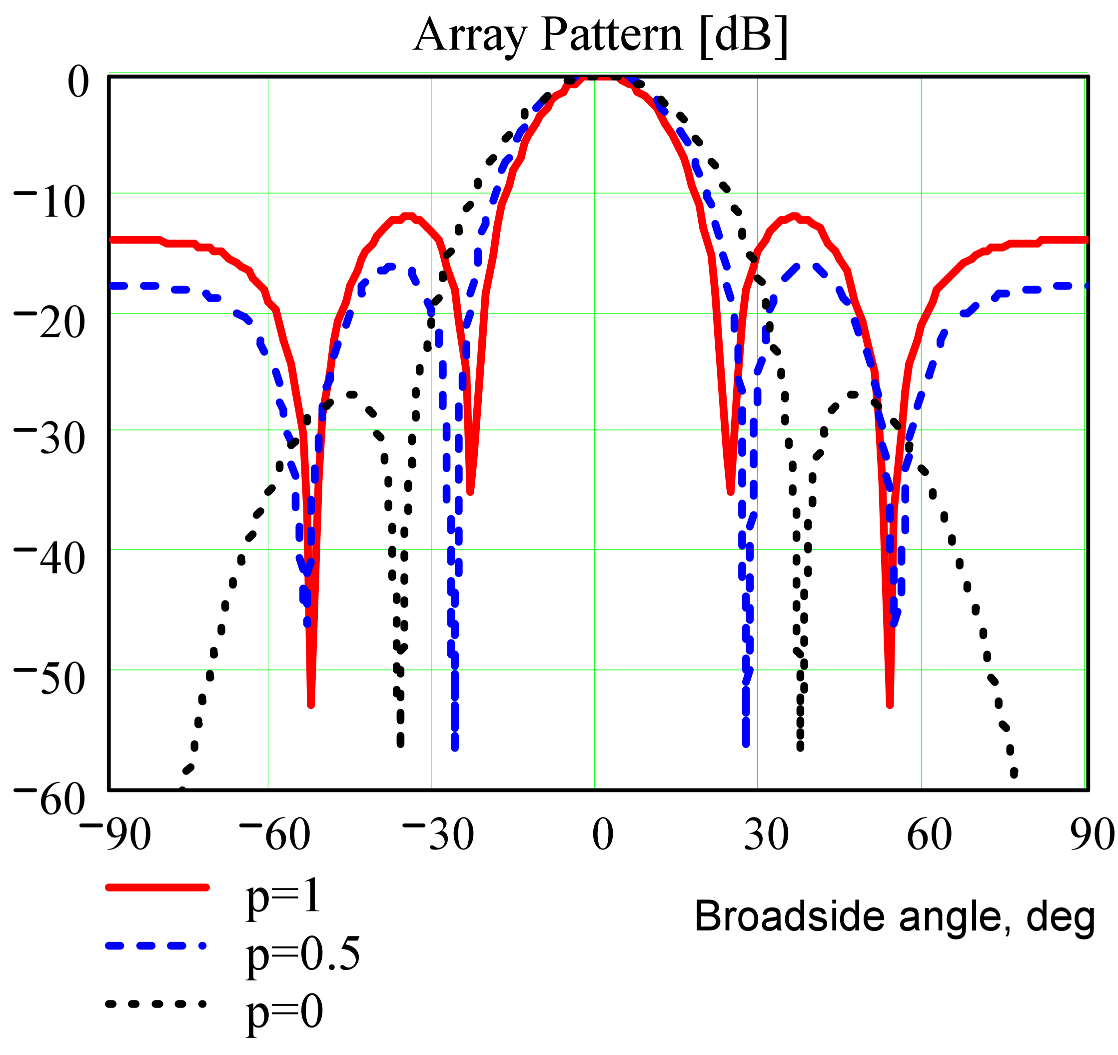
General principle: by decreasing weights at the array ends, we expand its main beam, but decrease side lobes (design trade off).

There are many other types of weighting to achieve specific goals.

Note: beam steering can be included as well.

Raised Cosine Weighting

Example: uniformly-spaced linear array with $N = 5;\ d = \lambda/2;$



Note that the sidelobe level decreases when decreasing p.

p=1 corresponds to the uniform weighting.

Other Weightings

Dolph-Chebychev weighting (minimum possible beamwidth for given (constant) sidelobe level). Makes use of the Chebychev polynomials.

Taylor weighting (constrains the maximum sidelobe level and gives decaying outer sidelobes).

Hamming weighting (place a null at the peak of first sidelobe).

Blackman-Harris weighting (place nulls at the peaks of first two sidelobes).

See Van Trees for more details and design procedures.

Null Steering and LMS Pattern Synthesis

Considering the following problem: find the best LMS (least mean square) approximation to a desired pattern, subject to a null constraints, i.e. the pattern must have nulls in given directions.

The desired and approximate patterns are[1]

$$F_d(\mathbf{k}) = \mathbf{w}_d^+ \mathbf{v}(\mathbf{k}), \qquad F(\mathbf{k}) = \mathbf{w}^+ \mathbf{v}(\mathbf{k}) \tag{5.2}$$

MS error between desired and approx. patterns is

$$\varepsilon = \int \left| F_d(\mathbf{k}) - F(\mathbf{k}) \right|^2 d\mathbf{k} = \left| \mathbf{w}_d - \mathbf{w} \right|^2 \tag{5.3}$$

Q.: prove it!

Constraint on main beam $\mathbf{w}^+ \mathbf{v}(\mathbf{k}_{mb}) = 1$,

i.e. the main beam direction is fixed to $\mathbf{k}_{mb}$ - distortionless constraint, keep the main beam fixed while steering the nulls. Any plane wave signal arriving at $\mathbf{k}_{mb}$ will pass without distortion.

Zero-order null constraint:

$$\mathbf{w}^+ \mathbf{v}(\mathbf{k}_i) = 0, \quad i = 1, 2, ..., M_0 \tag{5.4}$$

The patter nulls are formed at $\mathbf{k}_i$.

[1] Notations: bold capital (**K**) – matrices; bold lower case (**k**) – vectors; lower case regular (k) – scalars; $\mathbf{k}_i$ - i-th column of **K**.

Define the constraint matrix:

$$\mathbf{C}_0 = \left[\mathbf{v}(\mathbf{k}_1) \quad \mathbf{v}(\mathbf{k}_2) \quad \dots \quad \mathbf{v}(\mathbf{k}_{M_0}) \right] \qquad (5.5)$$

so that $\mathbf{w}^+\mathbf{C}_0 = \mathbf{0}^T$.

First-order null constraint:

$$\frac{d}{d\mathbf{k}} F(\mathbf{k})\Big|_{\mathrm{k}=\mathrm{k}_i} = \mathbf{w}^+\mathbf{d}_1(\mathbf{k})\Big|_{\mathrm{k}=\mathrm{k}_i} = 0, \quad i \in \Omega_1 \qquad (5.6)$$

1^{st}-order derivatives of the pattern are set to zero at some directions; Ω_1 is a subset of M_0 , and

$$\mathbf{d}_1(\mathbf{k}) = \frac{d}{d\mathbf{k}} \mathbf{v}(\mathbf{k}) \qquad (5.7)$$

For 1-D array, only one component of $\mathbf{k}$ (e.g. k_x if the array is located along OX axis) is used in (5.7) (explain why!). See Van Trees (sec. 3.7.2) for an example.

Similarly, the constraint matrix

$$\mathbf{C}_1 = \left[\mathbf{d}_1(\mathbf{k}_1) \quad \mathbf{d}_1(\mathbf{k}_2) \quad \dots \quad \mathbf{d}_1(\mathbf{k}_{M_1}) \right] \qquad (5.8)$$

We can also introduce higher-order constraints (i.e. second-order derivatives etc.).

The total constraint matrix:

$$\mathbf{C} = \left[\mathbf{C}_0 \quad \mathbf{C}_1 \right] \qquad (5.9)$$

Total number of constraints $M_c = M_0 + M_1$ must satisfy $M_c \le N - 1$. Q.: explain why!

Assume that $\mathbf{C}$ is not singular (if not, retain only independent constraints).

<u>Optimization problem:</u>

$$\text{minimize } \varepsilon = \left|\mathbf{w}_d - \mathbf{w}\right|^2 \text{ subject to } \mathbf{w}^+\mathbf{C} = 0 \qquad (5.10)$$

<u>Solution</u>: using Lagrange multipliers (see the Review of Optimization Theory for details); the goal function is

$$G = [\mathbf{w}_d - \mathbf{w}]^+ [\mathbf{w}_d - \mathbf{w}] + \mathbf{w}^+\mathbf{C}\boldsymbol{\lambda} + \boldsymbol{\lambda}^+\mathbf{C}^+\mathbf{w} \qquad (5.11)$$

where $\boldsymbol{\lambda}$ is a (vector) Lagrange multiplier.

Take the gradient (derivative with respect to a vector),

$$\nabla_{\mathbf{w}^+} G = \frac{\partial G}{\partial \mathbf{w}^+} = \mathbf{0} \qquad (5.12)$$

to obtain

$$\mathbf{w}_0 = \mathbf{w}_d - \mathbf{C} \cdot \boldsymbol{\lambda} \qquad (5.13)$$

Using $\mathbf{w}_0^+\mathbf{C} = 0$, we obtain:

$$\mathbf{w}_d^+\mathbf{C} - \boldsymbol{\lambda}^+\mathbf{C}^+\mathbf{C} = 0 \Rightarrow \boldsymbol{\lambda}^+ = \mathbf{w}_d^+\mathbf{C}\left[\mathbf{C}^+\mathbf{C}\right]^{-1} \qquad (5.14)$$

Substitute it to (5.13) to obtain,

$$\mathbf{w}_0 = (\mathbf{I}_N - \mathbf{P}_c)\mathbf{w}_d = \mathbf{P}_{c\perp}\mathbf{w}_d \qquad (5.15)$$

where $\mathbf{P}_\mathrm{c}$ is the projection matrix into the constraint space spanned by the columns of $\mathbf{C}$:

$$\mathbf{P}_\mathrm{c} = \mathbf{C}\left[\mathbf{C}^+\mathbf{C}\right]^{-1}\mathbf{C}^+ \qquad (5.16)$$

Note that

$$\mathbf{P}_c\mathbf{C} = \mathbf{C},\ \mathbf{P}_c\mathbf{P}_c = \mathbf{P}_c \text{ and } \mathbf{P}_c\mathbf{x} = \mathbf{0} \text{ if } \mathbf{x}^+\mathbf{C} = \mathbf{0}, \qquad (5.17)$$

i.e. $\mathbf{P}_c$ is indeed a projection matrix.

Q1: prove the properties of $\mathbf{P}_c$ above!

It follows from (5.15) that the optimum weight vector $\mathbf{w}_0$ is the desired weight $\mathbf{w}_d$ minus its projection into the constraint space $\mathbf{C}$.

Q2: how do we know that (5.15) gives the minimum and not the maximum of ε? Is the solution unique?

Another form (interpretation) of the optimum weights:

$$\mathbf{w}_0 = (\mathbf{I}_N - \mathbf{P}_c)\mathbf{w}_d = \mathbf{w}_d - \mathbf{C}\mathbf{a} \qquad (5.18)$$

where $\mathbf{a}$ is a $1 \times M_c$ vector (weight),

$$\mathbf{a} = \left[\mathbf{C}^+\mathbf{C}\right]^{-1}\mathbf{C}^+\mathbf{w}_d \qquad (5.19)$$

Q3: consider the ULA as in the example of Lec. 4, slide 3, and compare its pattern there with that obtained by using the weights in (5.15), for the same null directions and other parameters; assume that the desired pattern for (5.15) is that of uniform weights steered to broadside. What is the difference, if any, between 2 patterns? Plot them on the same graph.

It follows from (5.18) that the optimum weights are the desired weights minus a weighted sum of the constraint vectors.

The optimum pattern is

$$F_o(\mathbf{k}) = F_d(\mathbf{k}) - \mathbf{a}^+\mathbf{C}^+\mathbf{v}(\mathbf{k}) \tag{5.20}$$

For zero-order constraints, 2nd term is sum of conventional patterns steered at the directions of nulls (interferers),

$$F_o(\mathbf{k}) = F_d(\mathbf{k}) - \sum_{i=0}^{M_0} a_i F_c(\mathbf{k} - \mathbf{k}_m) \tag{5.21}$$

For a ULA,

$$F_c(\mathbf{k}\text{ - }\mathbf{k}_m) = \frac{\sin(\frac{N\pi d}{\lambda}(\cos\theta - \cos\theta_m))}{N\sin(\frac{\pi d}{\lambda}(\cos\theta - \cos\theta_m))} \tag{5.22}$$

The optimum pattern is the desired pattern minus a weighted sum of conventional patterns (uniform weighting) steered at the null directions.

This is true for arbitrary arrays as well.

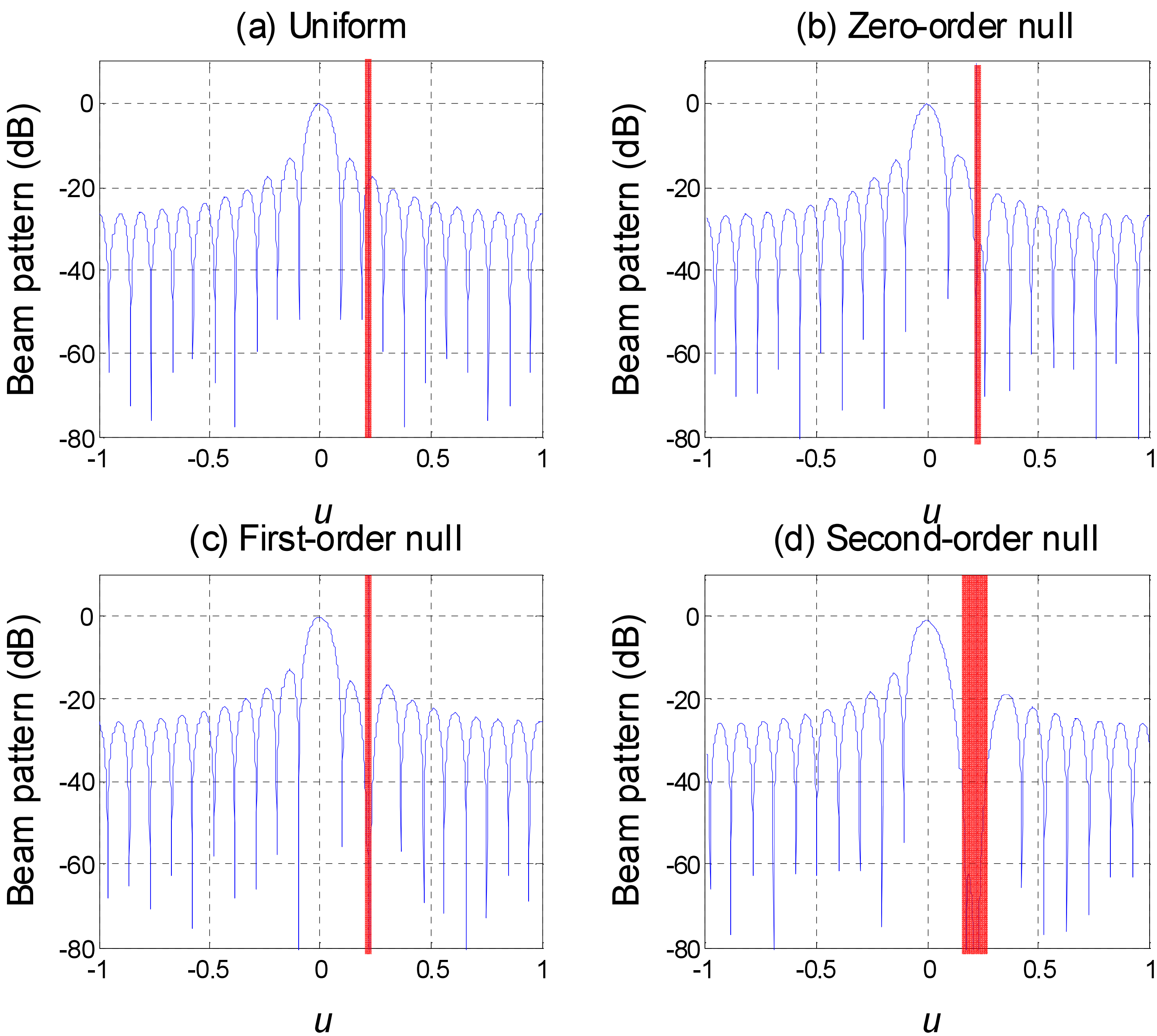
The resulting pattern error is

$$\varepsilon_0 = \mathbf{w}_e^+\mathbf{w}_e = |\mathbf{w}_e|^2 = \mathbf{w}_d^+\mathbf{P}_c\mathbf{w}_d \tag{5.23}$$

where $\mathbf{w}_e = \mathbf{w}_d - \mathbf{w}_0$.

Examples

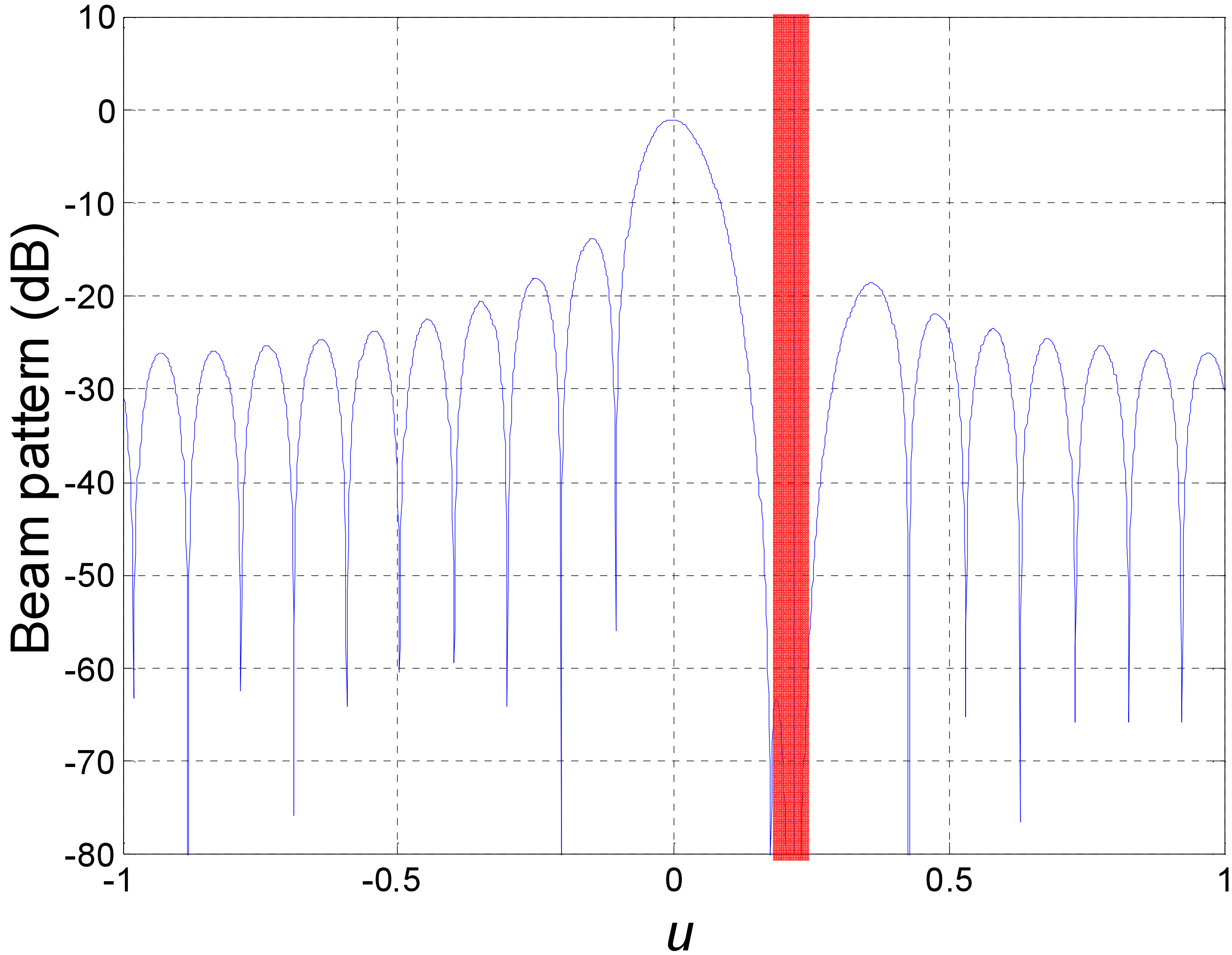
$$d = \lambda / 2,\ N = 21,\ u = \sin\bar{\theta} = \cos\theta$$



H.L. Van Trees, Optimum Array Processing, Wiley

Examples

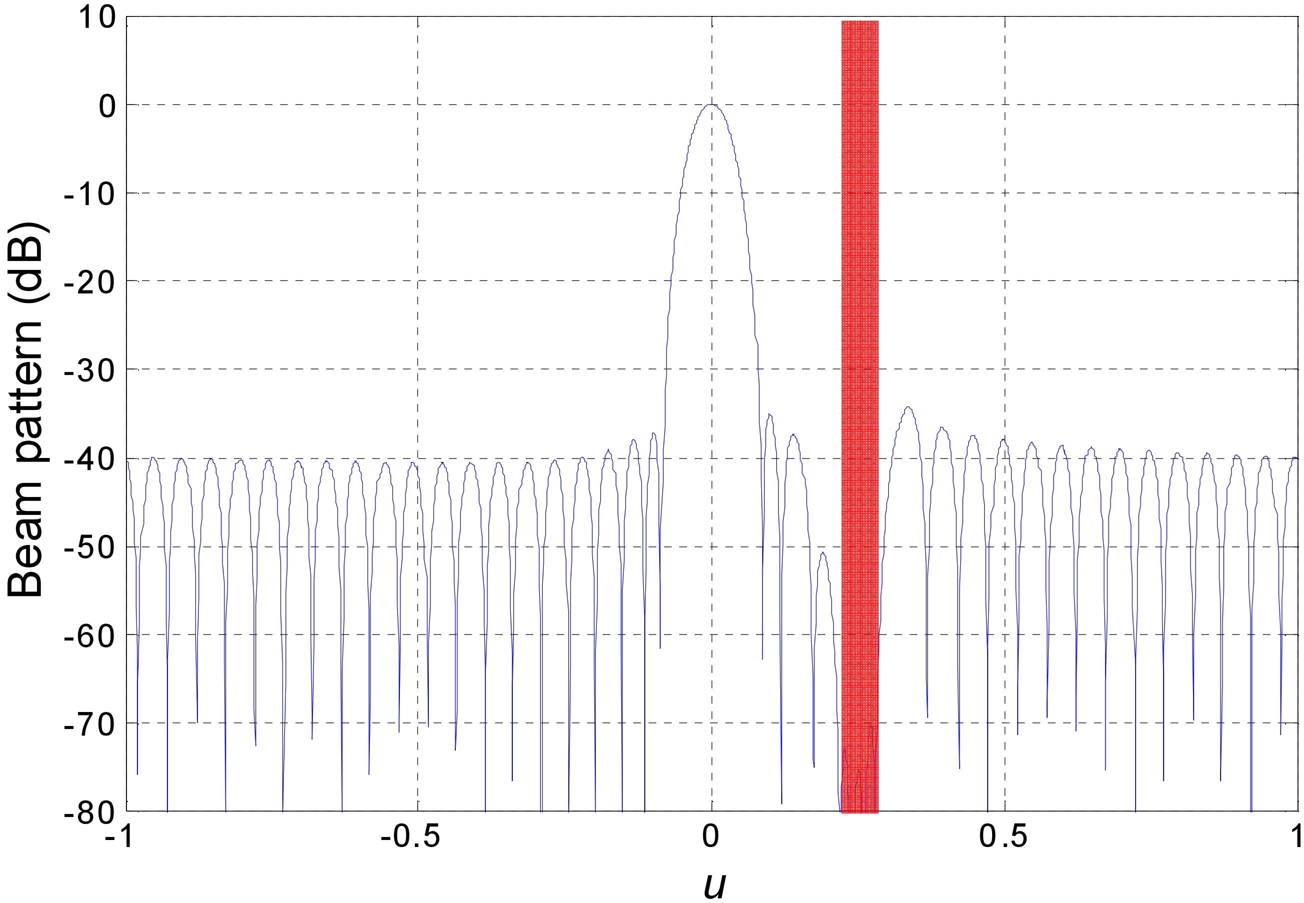
Initial sinc pattern with 3 nulls equispaced over the sector [0.18,0.26]



H.L. Van Trees, Optimum Array Processing, Wiley

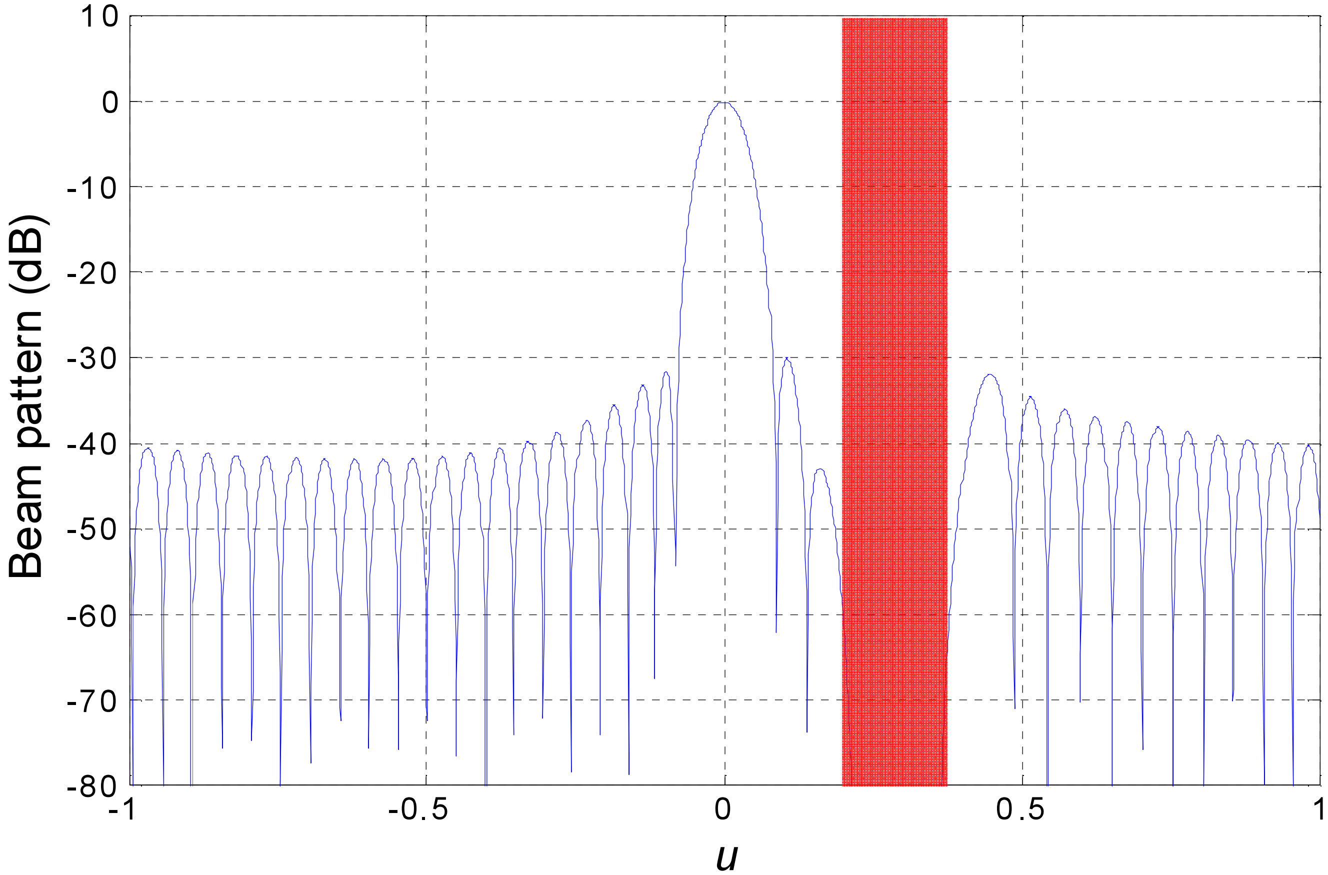
Examples

Initial 40 dB Chebyshev pattern with 4 nulls equispaced over the sector [0.22,0.28]



Examples

Initial 40 dB Chebyshev pattern with 8 nulls equispaced over the sector [0.22,0.36]



H.L. Van Trees, Optimum Array Processing, Wiley

Summary

- Synthesis of linear arrays. Various synthesis criteria.
- Sidelobe level reduction. Various weightings.
- Null steering for the desired pattern using LMS synthesis.
- Various constraints (zero-order, first-order, etc.).
- Synthesis procedure and optimum weights.

References:

1. H.L. Van Trees, Optimum Array Processing, Wiley, New York, 2002.
2. T.K. Moon, W.C. Stirling, Mathematical Methods and Algorithms for Signal Processing, Prentice Hall, 2000. (optimization – Ch. 18, complex derivatives/gradients – Appendix E).

- Homework: fill in the details in the derivations above. Do the examples yourself.