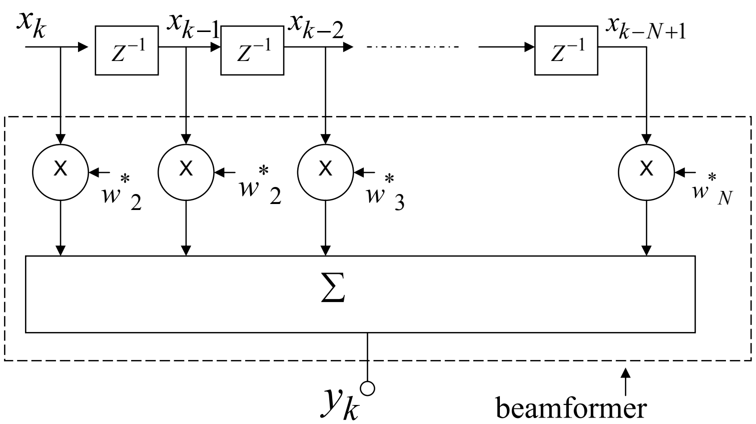
Adaptive Equalizer as a Beamformer



$$y_k = \sum_{i=1}^{N} w^*_i x_{k-i+1} = \mathbf{w}^+ \mathbf{x}_{(k)} \quad (1)$$

$$\mathbf{w} = [w_1, w_2, w_3 \ldots. w_N]^T \quad (2)$$

where z^{-1} is one symbol delay element, and k is temporal (symbol) index.

Optimum weights $\mathbf{w}$ should provide best estimate of the transmitted symbols.

While the beamformer is multiple input single output system, an equalizer is the single input system (serial rather than parallel input).

ISI Channel Model (discrete time)

The discrete time model of an ISI vector channel can be expressed in the following form,

$$\mathbf{x} = \mathbf{s}' + \xi \quad (3)$$

$\mathbf{s}'$ represents the required signal + ISI, and ξ is AWGN. Component-wise this can be represented as follows,

$$x_k = h_0 s_k + \underbrace{\sum_{i=1}^{L} h_i s_{k-i}}_{ISI} + \underbrace{\xi_k}_{noise} \quad (4)$$

$$s'_k = h_0 s_k + \underbrace{\sum_{i=1}^{L} h_i s_{k-i}}_{\text{ISI}} \quad (5)$$

where L is the memory (in symbols) of the channel.

Job of equalizer : find best estimate of s_k given $\mathbf{x}$, $\hat{s}_k(\mathbf{x}) = ?$

Linear equalizer = Beamformer

Best Linear equalizer = MMSE beamformer

Q. What is the physical reason for ISI ?

Q. What is the best (non linear) equalizer?

The equalizer output y_k serves as an estimate of s_k:

$$y_k = \hat{s}_k \tag{6}$$

The estimation error is

$$\varepsilon_k = \hat{s}_k - s_k = y_k - s_k \tag{7}$$

Further, we drop index k for clarity.

MMSE equalizer = MMSE beamformer: minimize MSE,

$$\begin{aligned} MSE &= \overline{|\varepsilon|^2} = \overline{|y-s|^2} \\ &= \overline{(y-s)(y-s)^+} \\ &= \overline{|y|^2} + \overline{|s|^2} - \overline{sy^+} - \overline{ys^+} \\ &= \mathbf{w}^+ \underbrace{\overline{\mathbf{x}\mathbf{x}^+}}_{\mathbf{R}_x} \mathbf{w} + \sigma_s^2 - \underbrace{\overline{s\mathbf{x}^+}}_{\mathbf{r}_{sx}^+} \mathbf{w} - \mathbf{w}^+ \underbrace{\overline{\mathbf{x}s^*}}_{\mathbf{r}_{sx}} \end{aligned} \tag{8}$$

$$\frac{\partial \mathrm{MSE}}{\partial \mathbf{w}} = \mathbf{w}^+ \mathbf{R}_x - \mathbf{r}_{sx}^+ = \mathbf{0} \Rightarrow \mathbf{w}_0^+ = \mathbf{r}_{sx}^+ \mathbf{R}_x^{-1} \tag{9}$$

Compare to the MMSE beamformer:

$$\mathbf{R}_x = \mathbf{S}_x; \; \underbrace{s = x_s;}_{\text{key!}} \; \mathbf{r}_{sx} = \overline{\mathbf{x} x_s^*} = \mathbf{v}_s \tag{10}$$

Key Property:

$$MSE = MMSE + (\mathbf{w} - \mathbf{w}_0)^+ \mathbf{R}_x (\mathbf{w} - \mathbf{w}_0) \tag{11}$$

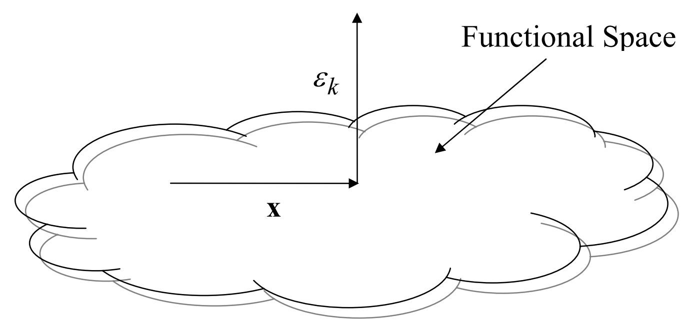
$$MMSE = \sigma_s^2 - \mathbf{r}_{sx}^+ \mathbf{R}_x^{-1} \mathbf{r}_{sx} \tag{12}$$

Q. Prove it!

Orthogonality Principle

$$\overline{\mathbf{x}\varepsilon_k^*} = \overline{\mathbf{x}\mathbf{x}^+}\mathbf{w}_0 - \overline{\mathbf{x}s_k^*} = \mathbf{R}_x \mathbf{R}_x^{-1} \mathbf{r}_{sx} - \mathbf{r}_{sx} = \mathbf{0} \tag{13}$$

which means that $\mathbf{x}$ and ε_k are not correlated ($\mathbf{x}$ cannot be used to reduce ε_k further).



In other words, $\mathbf{x}$ and ε_k are statistically orthogonal.

This is a very general and important principle (for MMSE).

The orthogonality principle can be derived directly from the basic optimum condition,

$$\begin{aligned}\frac{\partial \mathrm{MSE}}{\partial \mathbf{w}} &= \frac{\partial}{\partial \mathbf{w}} \overline{(\mathbf{w}^+\mathbf{x} - s)(\mathbf{x}^+\mathbf{w} - s^*)} \\ &= \overline{(\underbrace{\mathbf{w}^+\mathbf{x} - s}_{\varepsilon})\mathbf{x}^+} = \overline{\varepsilon \mathbf{x}^+} = 0\end{aligned} \tag{14}$$