

Review of Linear Systems

- Linear system -> the superposition principle holds:

$$\mathbf{L}[a_1x_1(t) + a_2x_2(t)] = a_1\mathbf{L}[x_1(t)] + a_2\mathbf{L}[x_2(t)]$$

- Time-invariant system -> shift in time does not change the response:

$$y(t) = \mathbf{L}[x_1(t)] \rightarrow y(t - t_0) = \mathbf{L}[x_1(t - t_0)], \forall t_0$$

i.e., the operator $\mathbf{L}[\]$ does not change in time

- Linear time-invariant (LTI) system -> both, linear and time-invariant

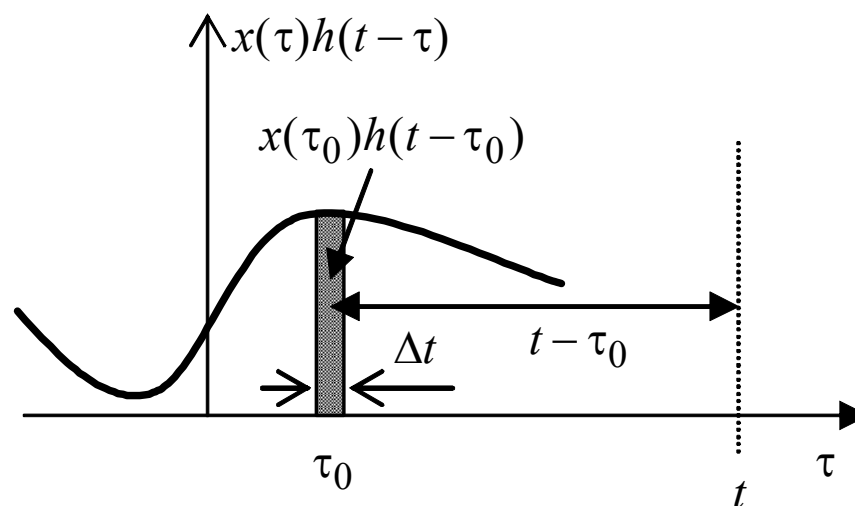
Response of LTI System

- If the input is a Dirac delta function, the output is impulse response:

$$h(t) = \mathbf{L}[\delta(t)]$$

- If the input is $x(t)$, the output is a convolution integral:

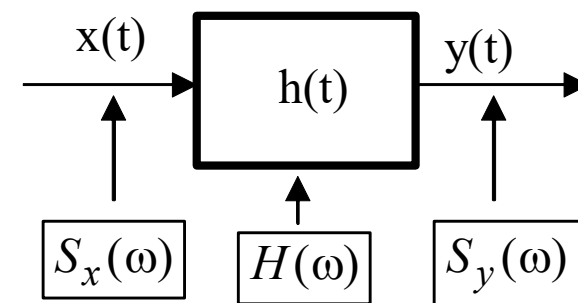
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$



Transfer Function

- Convolution theorem in frequency domain:

$$y(t) = x(t) * h(t) \leftrightarrow S_y(f) = S_x(f)H(f)$$



- Transfer function (frequency response):

$$H(f) = \frac{S_y(f)}{S_x(f)}$$

$$h(t) \leftrightarrow H(f) = \mathbf{L}[1]$$

- Amplitude response & phase response:

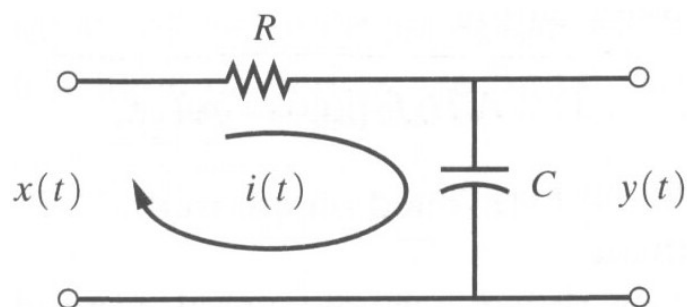
$$H(f) = |H(f)|e^{j\varphi(f)} = |H(f)|\angle\varphi(f)$$

$$\varphi(f) = \tan^{-1} \left(\frac{\text{Im}\{H(f)\}}{\text{Re}\{H(f)\}} \right)$$

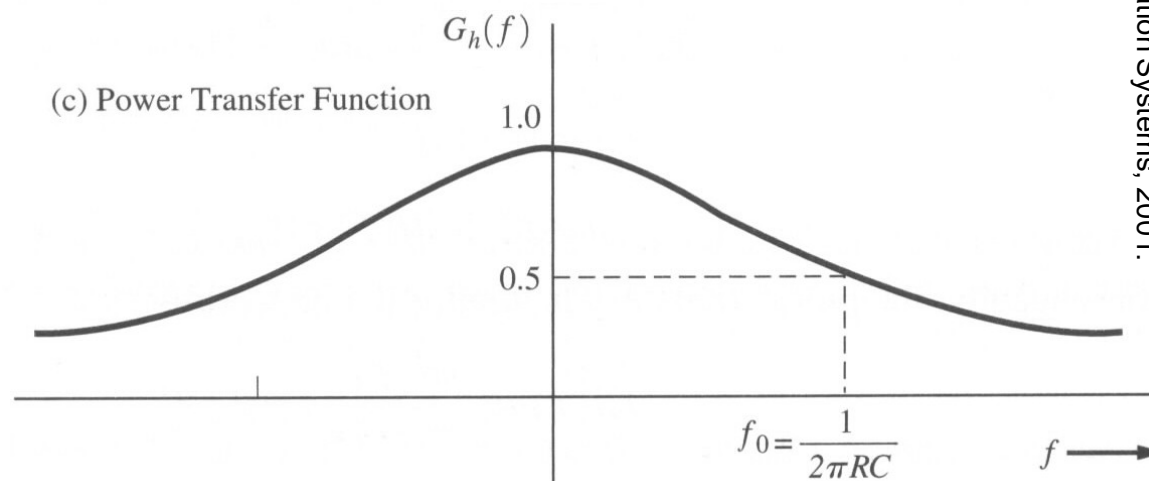
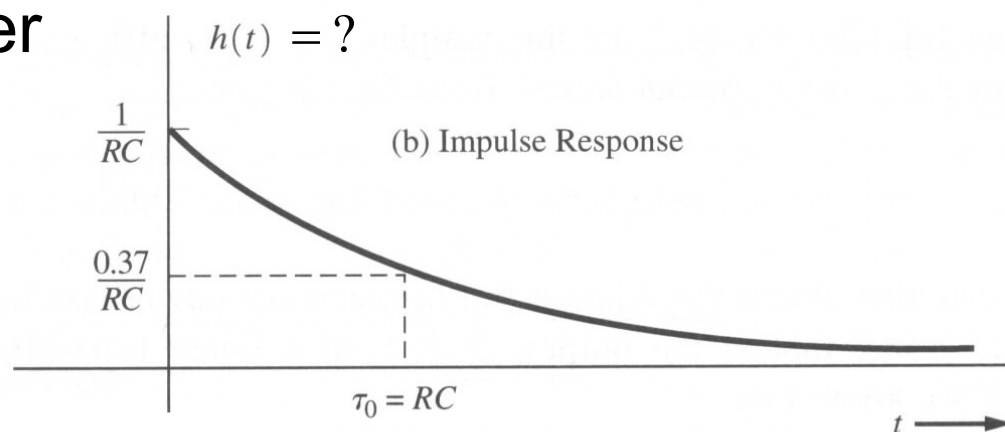
Power Transfer Function

- Power transfer function: $G(f) = \frac{P_y(f)}{P_x(f)} = |H(f)|^2$

- Example 2.14: RC filter



(a) RC Low-Pass Filter



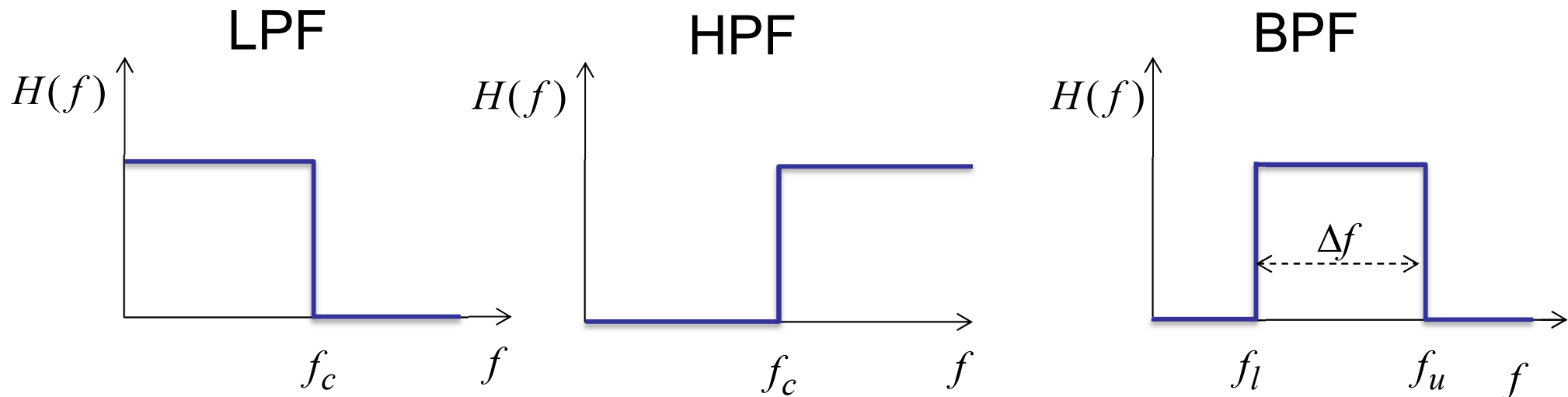
Derive it.

$$H(f) = \frac{1}{1 + jf / f_0}$$

$$G(f) = \frac{1}{1 + (f / f_0)^2}$$

Ideal (“brick-wall”) Filters

- low-pass filter (LPF)
- high-pass filter (HPF)
- band-pass filter (BPF)



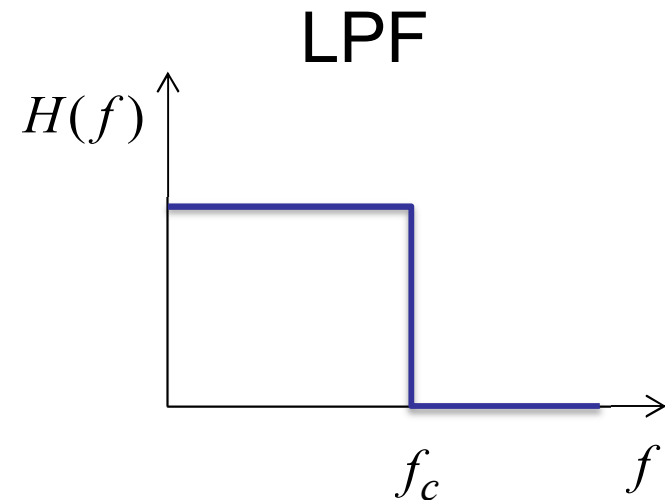
Ideal Filters: LPF

- Ideal (“brick-wall”) low-pass filter (LPF)
 - Frequency response:

$$H(f) = \begin{cases} 1, & |f| \leq f_c \\ 0, & |f| > f_c \end{cases}$$

- impulse response:

$$h(t) = ?$$



Ideal Filters: HPF, BPF

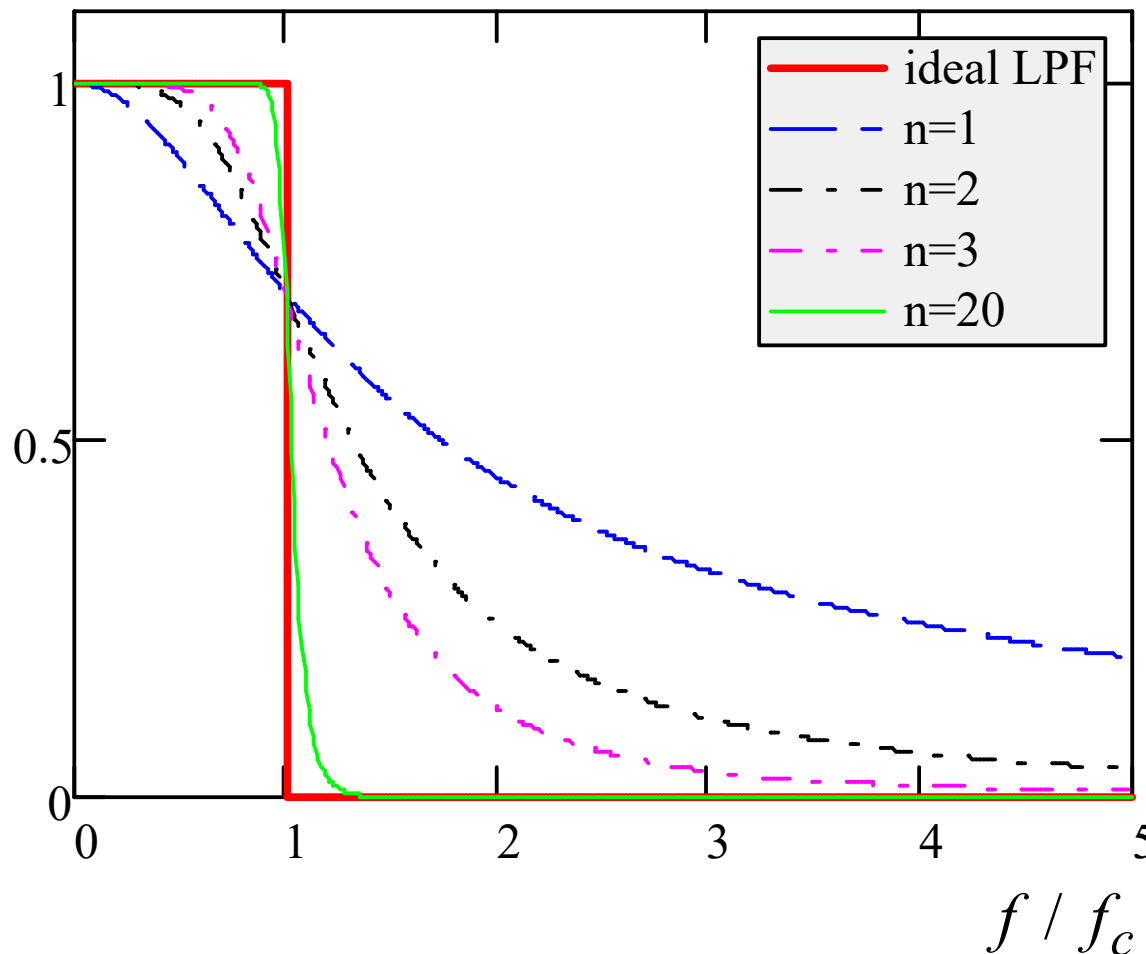
- HPF: $H(f)=?$ $h(t)=?$
- BPF: $H(f)=?$ $h(t)=?$

- How to build BPF using LPF and HPF?
 - suggest at least 2 ways

- Practical: measured in Lab 1

Practical Filters: Butterworth

- Frequency response: $|H_n(f)| = \frac{1}{\sqrt{1 + (f / f_c)^{2n}}}$



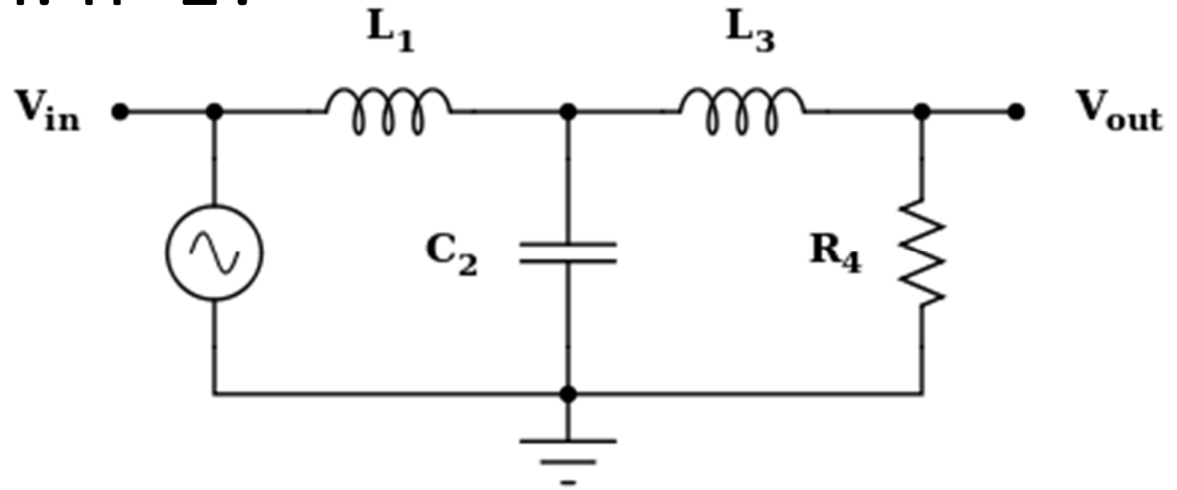
Q: show that

$$\lim_{n \rightarrow \infty} |H_n(f)| = H_{\text{LPF}}(f)$$

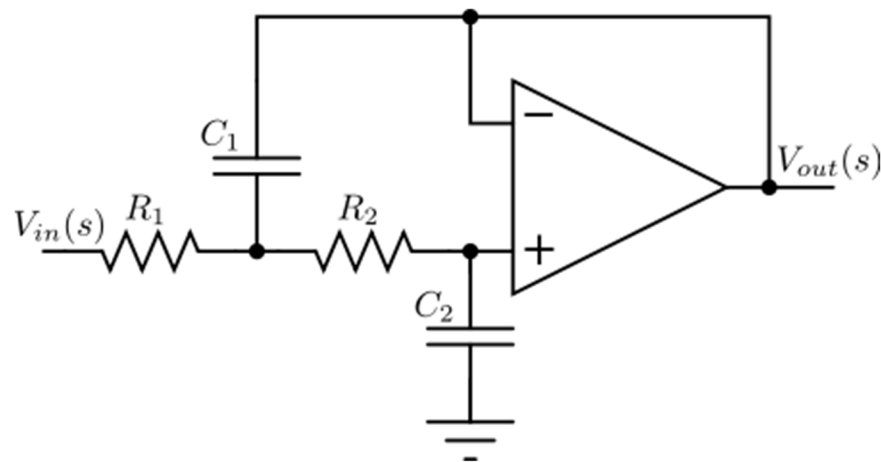
Practical Filters: Butterworth

- Implementation: $n=1$?

- $n=3$, passive

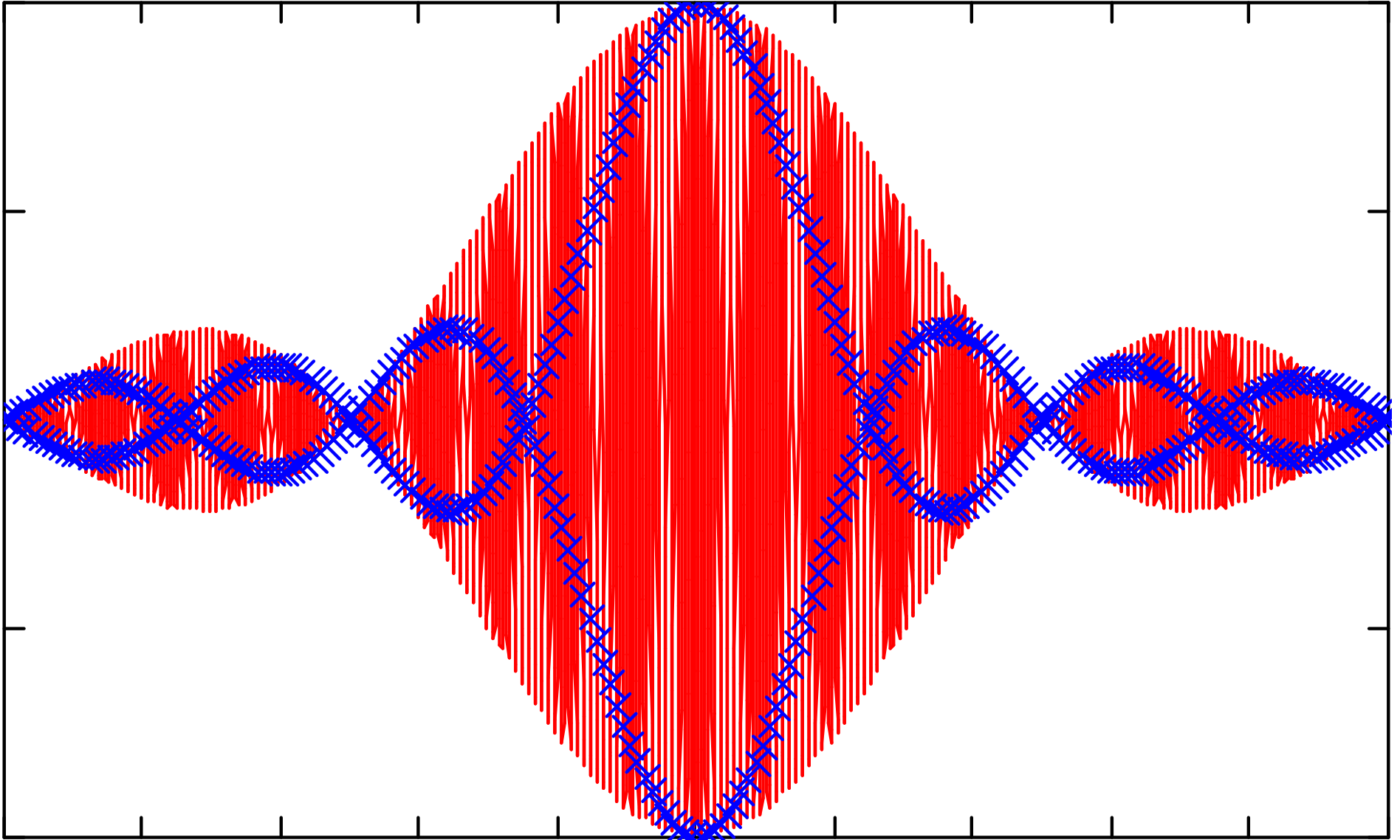


- $n=2$, active



Adopted from "Wikipedia: Butterworth filter"

???



Signal Bandwidth

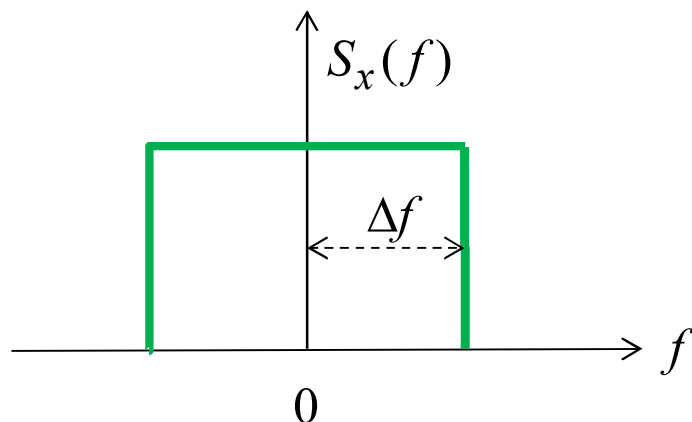
- Defined for positive frequencies only.
- Informal: a frequency band over which a substantial (or all) signal power is concentrated
- Absolute bandwidth: for band-limited signals, the minimum frequency band over which the spectrum is not zero. For all other frequencies, the spectrum must be zero:

$$S_x(f) = 0 \quad \forall f \notin [f_{\min}, f_{\max}] : \Delta f = f_{\max} - f_{\min}$$

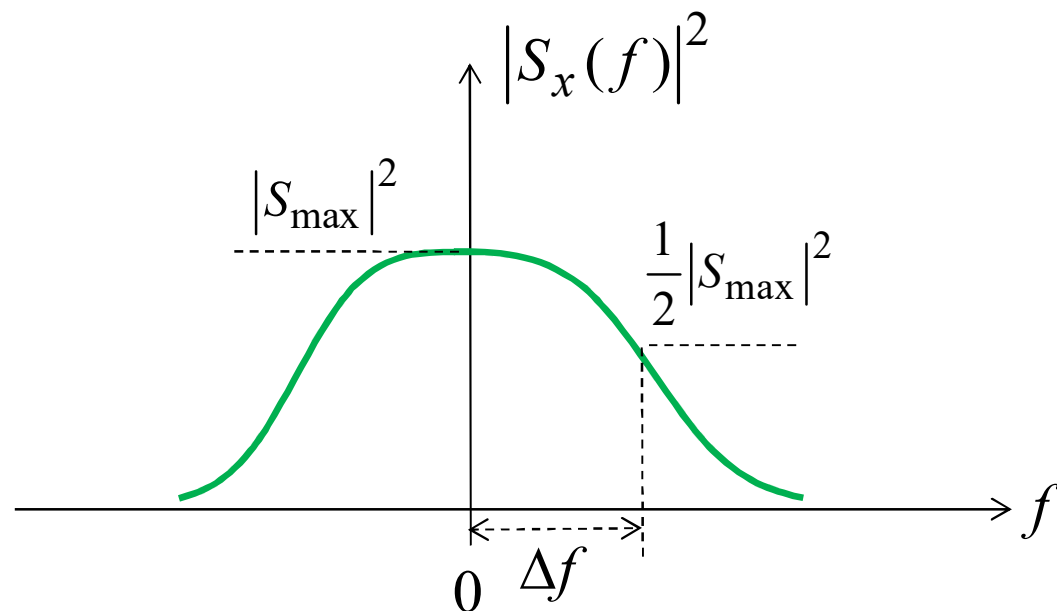
- 3 dB (half-power) bandwidth: frequency band where PSD (or ESD) is not lower than -3 dB with respect to maximum
- Zero-crossing bandwidth: frequency band limited by 1st zero(s) in the spectrum.

Signal Bandwidth (baseband/lowpass)

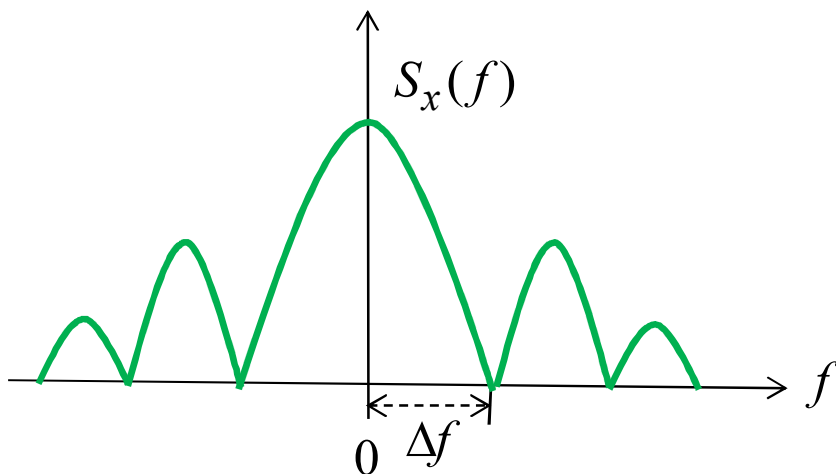
absolute bandwidth



3 dB bandwidth



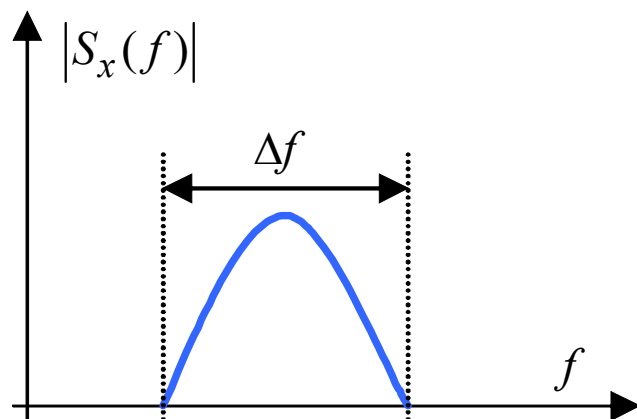
zero-crossing bandwidth



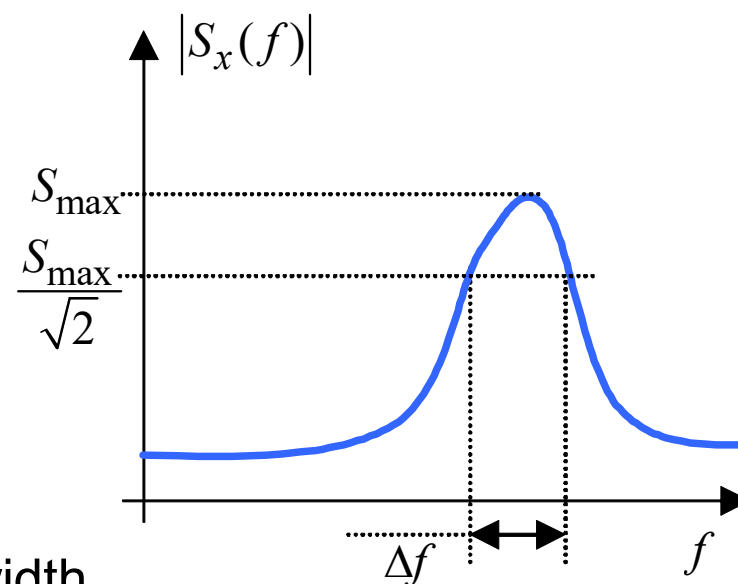
$$f_{\min} = 0, \quad \Delta f = f_{\max}$$

Signal Bandwidth (bandpass/RF)

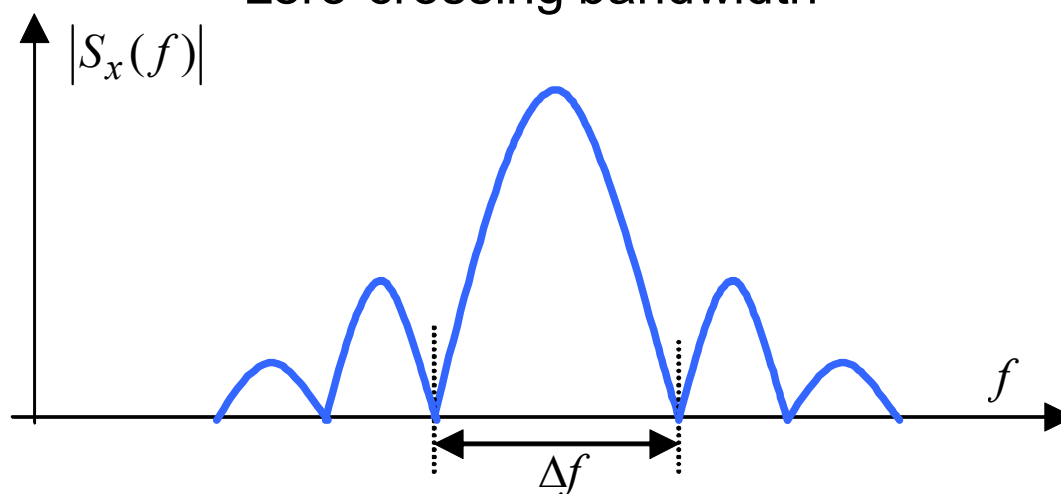
absolute bandwidth



3 dB bandwidth



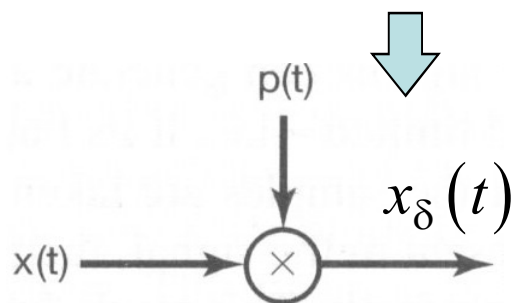
zero-crossing bandwidth



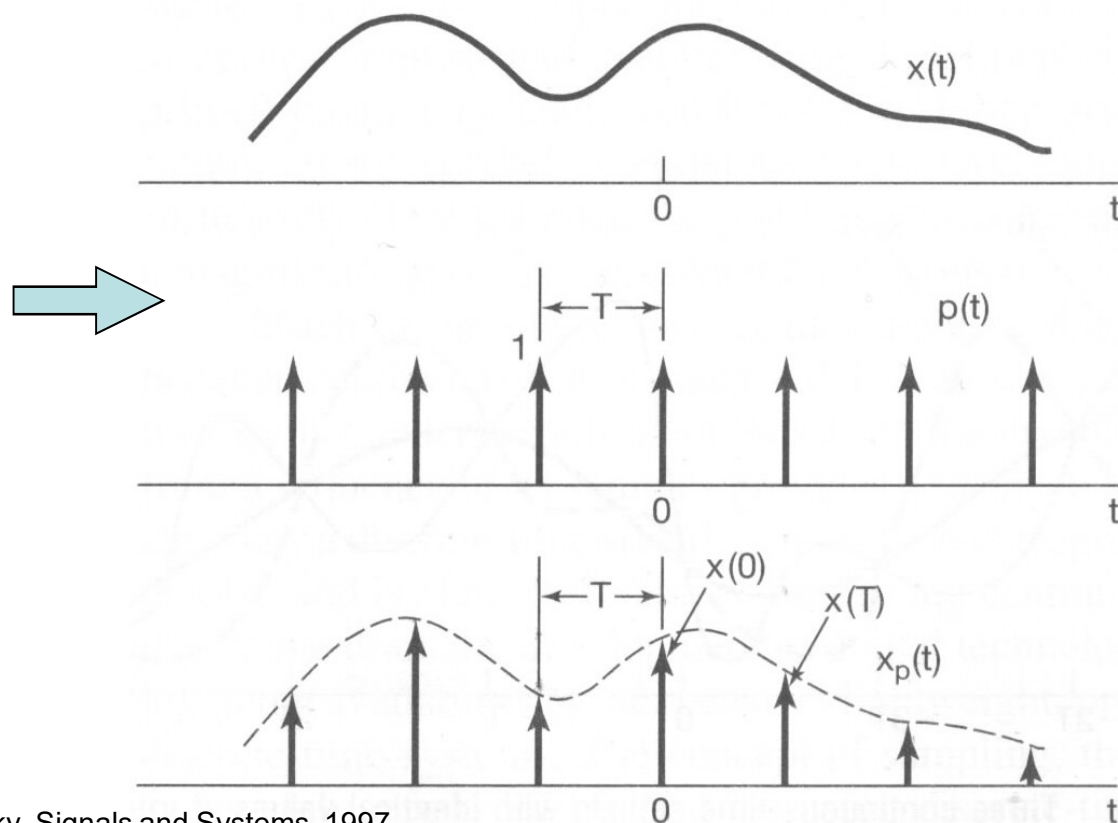
Sampling Theorem

- The sampling theorem is one of the most important results. A bridge between digital and analog worlds. DSP is based on it.
- Intuitive explanation: interpolation.
- Signal sampling:

$$x_{\delta}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.



Sampling Theorem: Conditions

1. A continuous bandlimited signal $x(t)$:

$$S_x(f) = 0, \forall |f| > f_{\max} \leftrightarrow x(t) = \int_{-f_{\max}}^{f_{\max}} S_x(f) e^{j2\pi ft} df$$

2. Absolutely-integrable FT (power-type OK):

$$\int_{-f_{\max}}^{f_{\max}} |S_x(f)| df < \infty$$

- (2) holds if energy-type:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-f_{\max}}^{f_{\max}} |S_x(f)|^2 df = E_x < \infty$$

The Sampling Theorem

If $x(t)$ is a continuous bandlimited signal with absolutely-integrable FT, then

1. $x(t)$ is uniquely determined, for any t , by its samples

$$x(nT_s), n = 0, \pm 1, \pm 2, \dots, T_s \leq 1 / (2f_{\max})$$

T_s = sampling interval.

2. $x(t)$ can be reconstructed from its samples as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right)$$

$f_s = 1/T_s \geq 2f_{\max}$ is the sampling frequency;

$2f_{\max}$ is the Nyquist frequency (the min. possible sampling frequency);

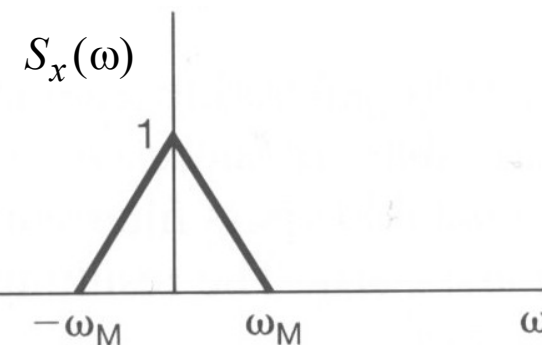
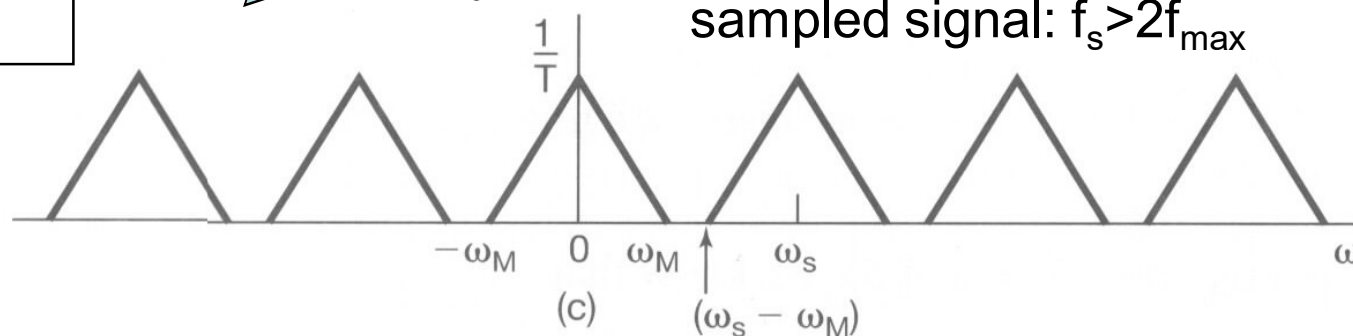
$f_s = 2f_{\max}$ is allowed if no singularity at $f = f_{\max}$

Sampling in Frequency Domain

$$x_{\delta}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

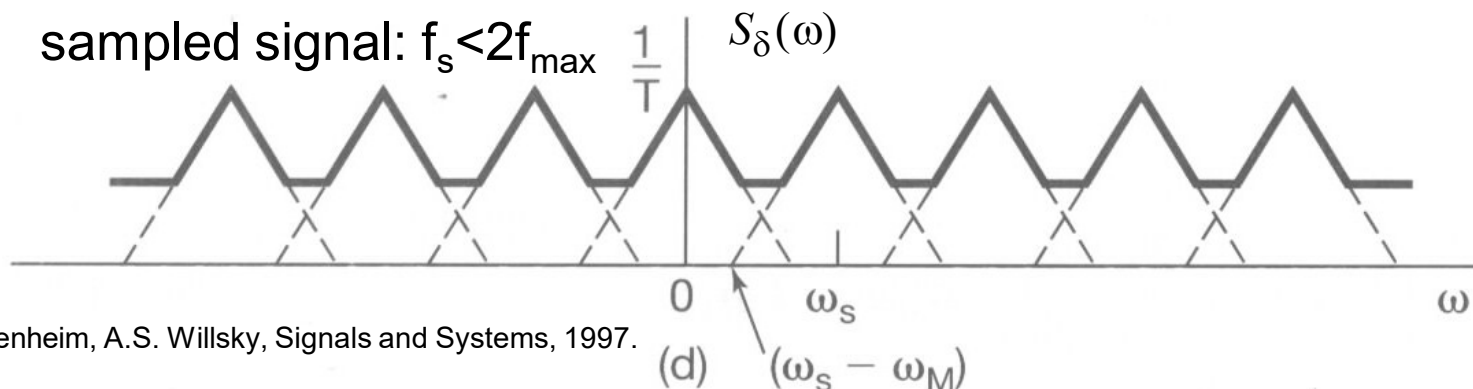
$$\begin{aligned} x_{\delta}(t) &\leftrightarrow S_x(\omega) * S_{\delta}(\omega) \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} S_x(\omega - n\omega_s) \end{aligned}$$

original signal

 $S_{\delta}(\omega)$ sampled signal: $f_s > 2f_{\max}$ 

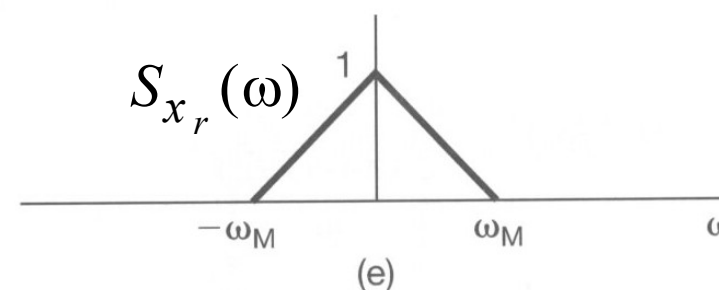
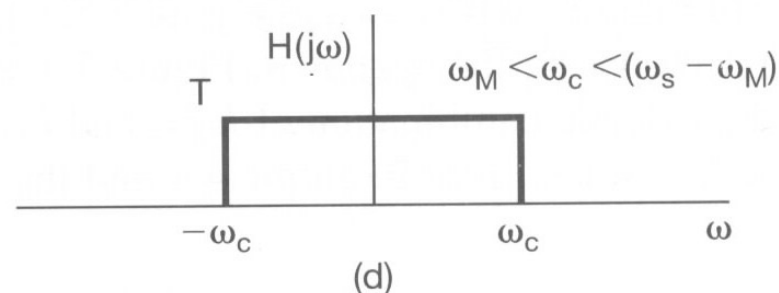
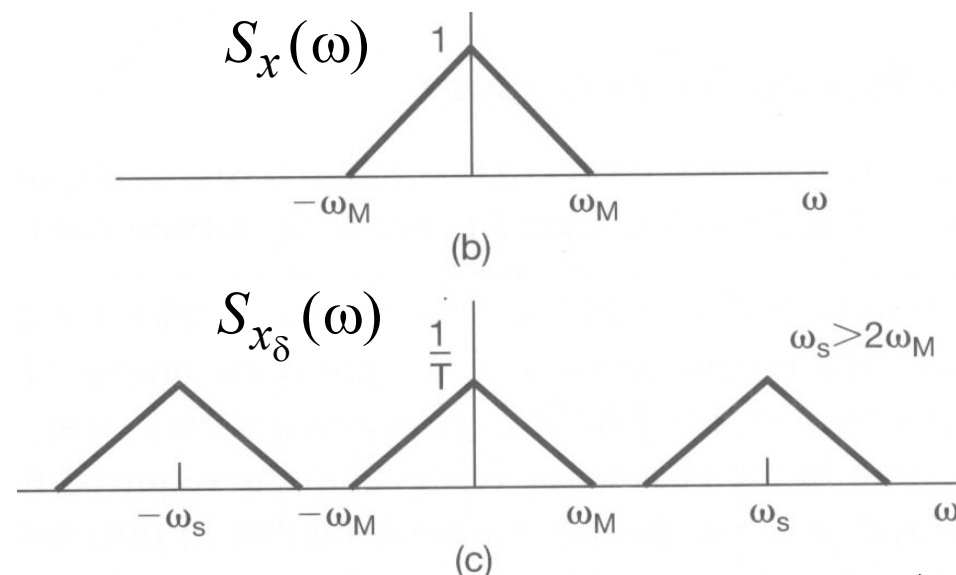
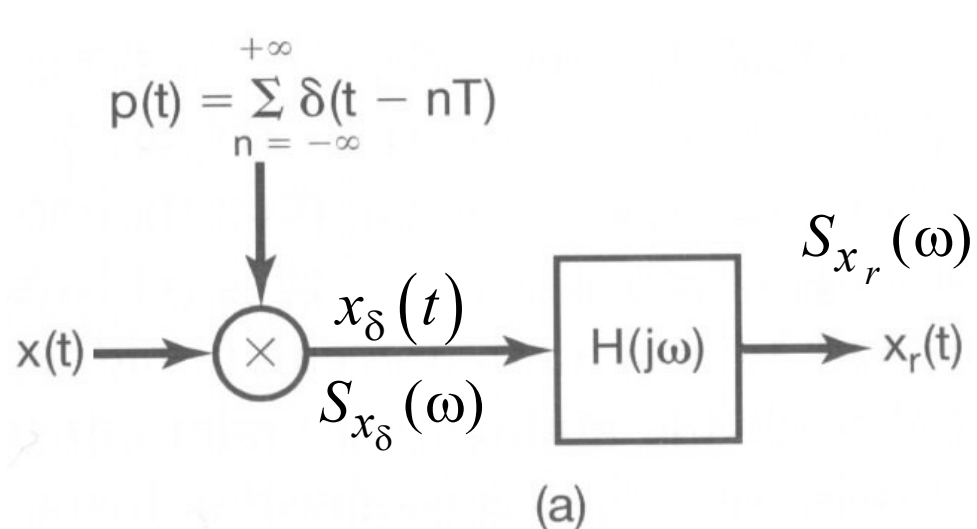
Nyquist
frequency:

$$f_{s,\min} = 2f_{\max}$$

sampled signal: $f_s < 2f_{\max}$ 

A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

Reconstruction in Frequency Domain



- sample the signal with $f_s > 2f_{\max}$
- use low-pass filter with $f_{\max} < f_c < f_s - f_{\max}$ to recover original signal

Sampling Theorem: Consequences

If $x(t)$ is a continuous bandlimited energy signal, then

1. Its energy is determined by its samples:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = T_s \sum_{n=-\infty}^{\infty} |x(nT_s)|^2, \quad T_s \leq 1 / (2f_{\max})$$

2. and

$$\int_{-\infty}^{\infty} x(t) dt = T_s \sum_{n=-\infty}^{\infty} x(nT_s)$$

3. If $y(t)$ also satisfies the conditions, then

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = T_s \sum_{n=-\infty}^{\infty} x(nT_s)y^*(nT_s)$$

Sampling Power-Type Signals

If $x(t)$ is a continuous bandlimited power-type signal, then

1. Its power is determined by its samples:

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(nT_s)|^2, \quad T_s < 1/(2f_{\max})$$

2. and

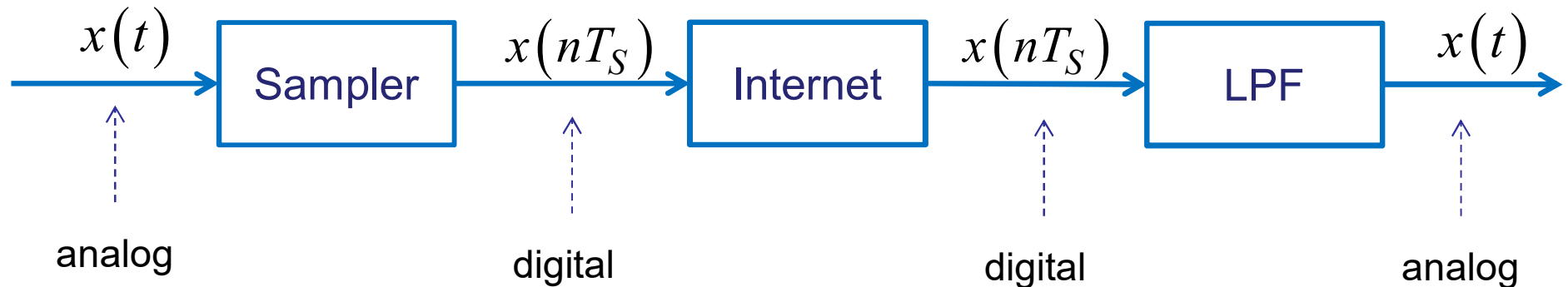
$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x(nT_s)$$

3. If $y(t)$ also satisfies the conditions, then

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)y^*(t) dt = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x(nT_s)y^*(nT_s)$$

Sampling Theorem and the Internet

(somewhat simplified: no quantizing yet)



$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_S) \operatorname{sinc}\left(\frac{t}{T_S} - n\right)$$

Sampling in Practice

- Approximately band-limited signals + LPF

$$S_x(f) \approx 0, \forall |f| > f_{\max} \leftrightarrow x(t) \approx \int_{-f_{\max}}^{f_{\max}} S_x(f) e^{j2\pi ft} df$$

- Approximate recovery from a finite number of samples over a finite time interval:

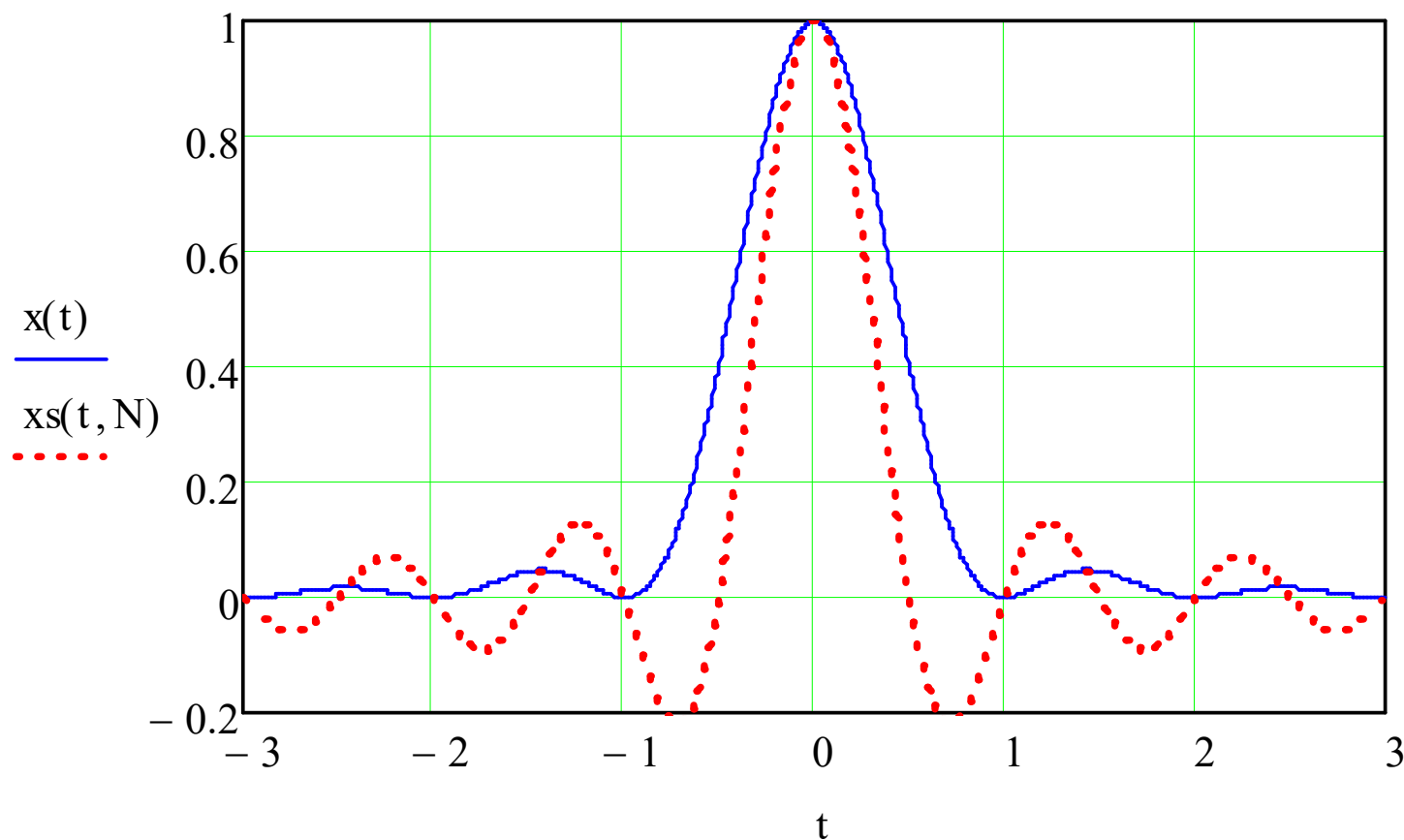
$$x(t) \approx x_N(t) = \sum_{n=-N}^N x(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right), \quad |t| \leq NT_s$$

- Approximation errors: must be small,

$$|x_N(t) - x(t)| \leq \varepsilon, \quad |t| \leq NT_s$$

Sampling $\text{sinc}^2(t)$

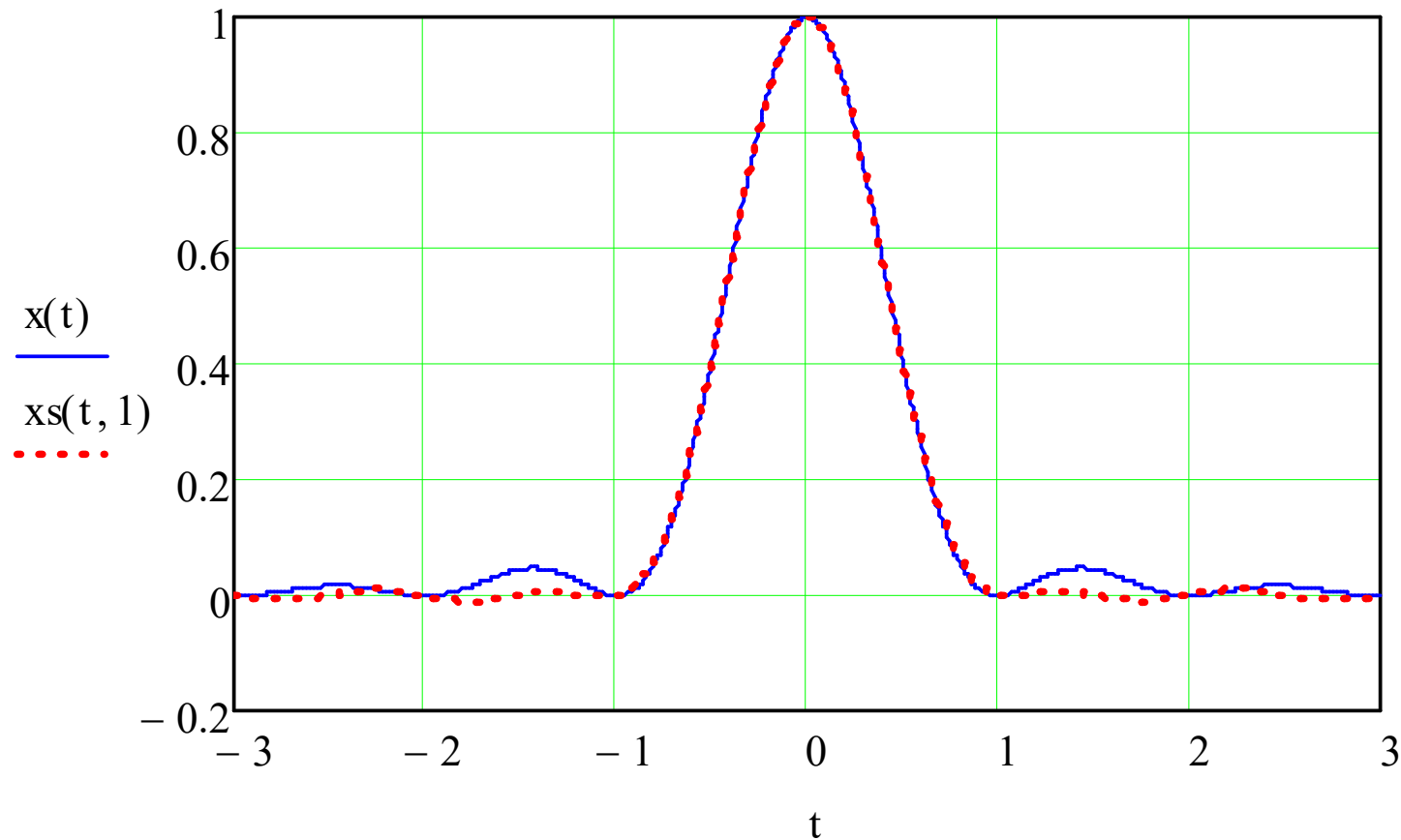
$$N_s = 2N + 1 = 1, \quad T_s = ?$$



$$x_s(t, N) = \sum_{n=-N}^N x(nT_s) \text{sinc}\left(\frac{t}{T_s} - n\right)$$

Sampling $\text{sinc}^2(t)$

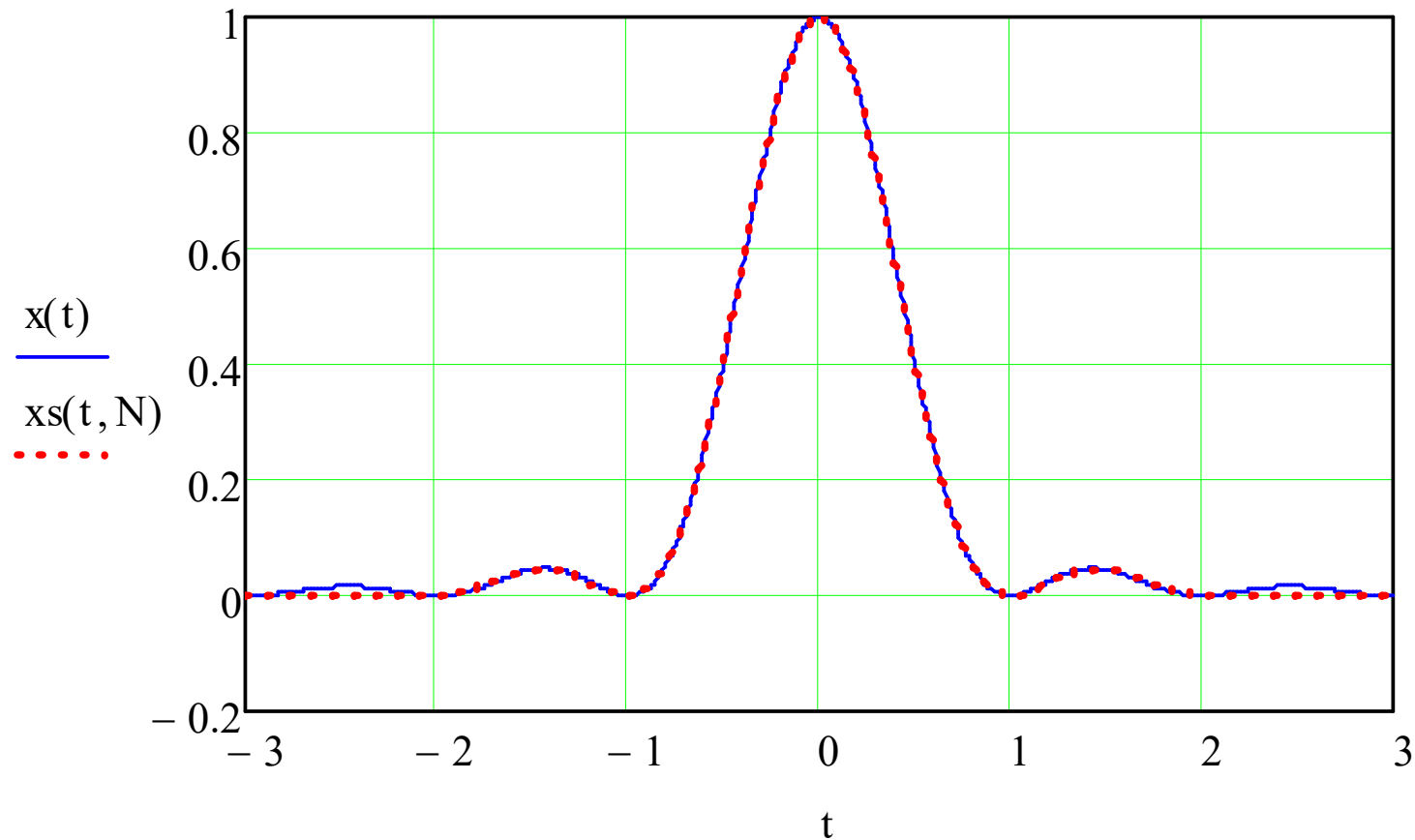
$$N_s = 2N + 1 = 3$$



$$x_s(t, N) = \sum_{n=-N}^N x(nT_s) \text{sinc}\left(\frac{t}{T_s} - n\right)$$

Sampling $\text{sinc}^2(t)$

$$N_s = 2N + 1 = 7$$



$$x_s(t, N) = \sum_{n=-N}^N x(nT_s) \text{sinc}\left(\frac{t}{T_s} - n\right)$$

Dimensionality Theorem

- A real signal of bandwidth B can be *approximately* recovered from

$$N = 2BT$$

samples over a time interval T provided that $BT \gg 1$

- This is a foundation for digital communications, signal processing, Internet, etc.
- If not band-limited?
- Sampling in practice: $f_s = 2B + \Delta f$
- **Example:** 1h of HiFi music, $N=?$

On Time/Bandlimited Signals

- An absolutely bandlimited signal cannot be time limited and vice versa.
- Engineering implications?

Sampling Theorem

Kotelnikov (1933)



Shannon (1948)



Whittaker (1915)



[1] V.A. Kotelnikov, On the carrying capacity of the ether and wire in telecommunications, The First All-Union Conference on Questions of Communication, Izd. Red. Upr. Svyazi RKKA (in Russian), 1933.

[2] C.E. Shannon, A Mathematical Theory of Communication, Bell System Technical Journal, Oct. 1948.

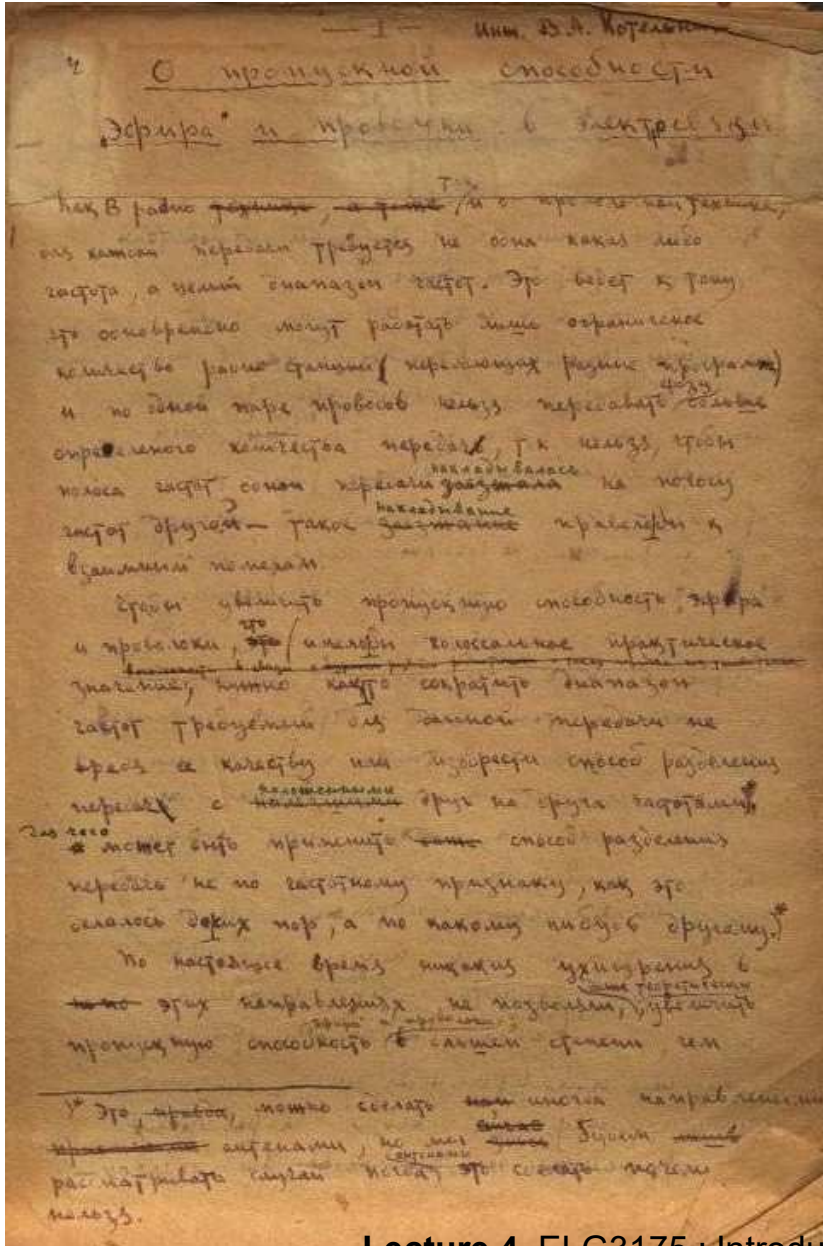
[3] E.T. Whittaker, On the Functions Which are Represented by the Expansions of the Interpolation Theory, Proc. Royal Soc. Edinburgh, 1915.

adopted from <https://en.wikipedia.org/>

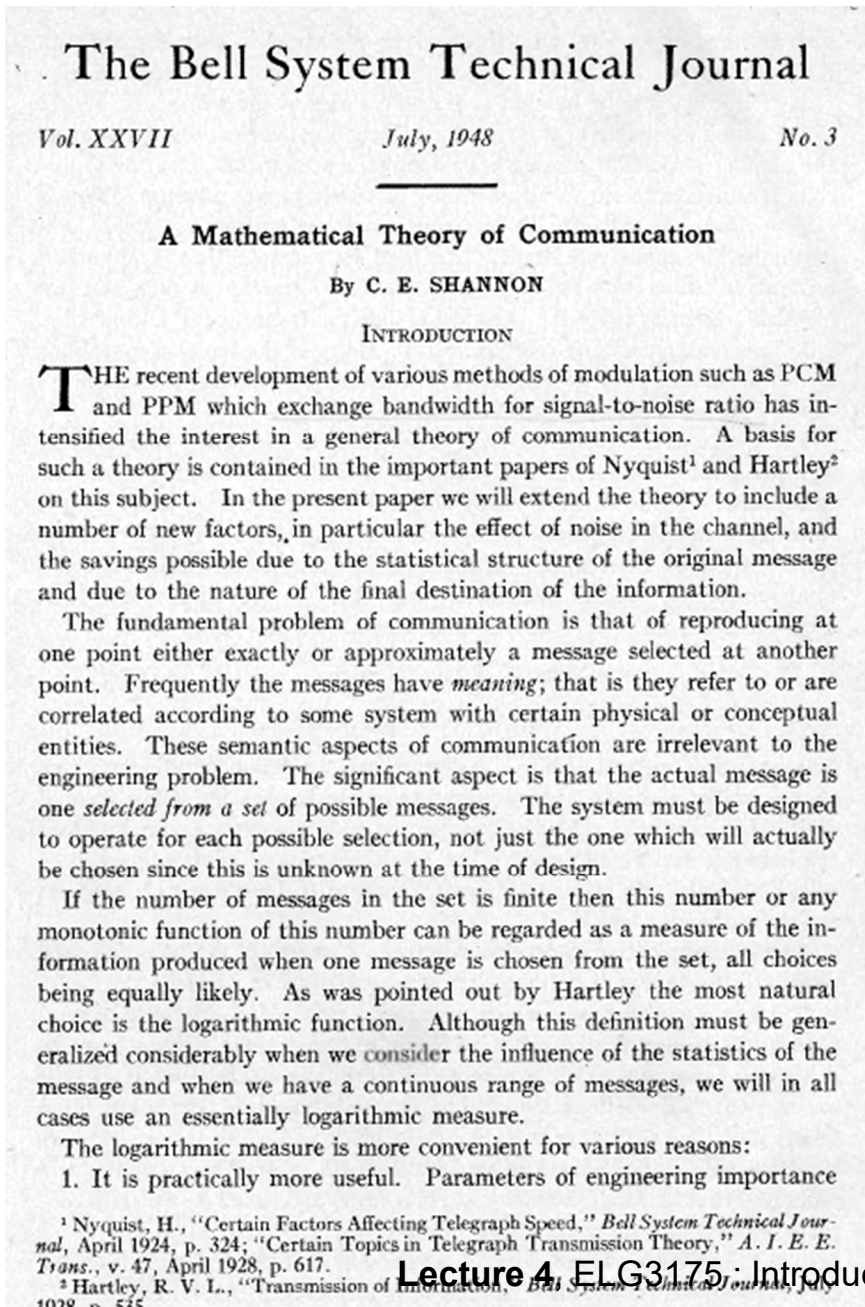
Vladimir Kotelnikov

Born: 6 Sep. 1908, Kazan, USSR.

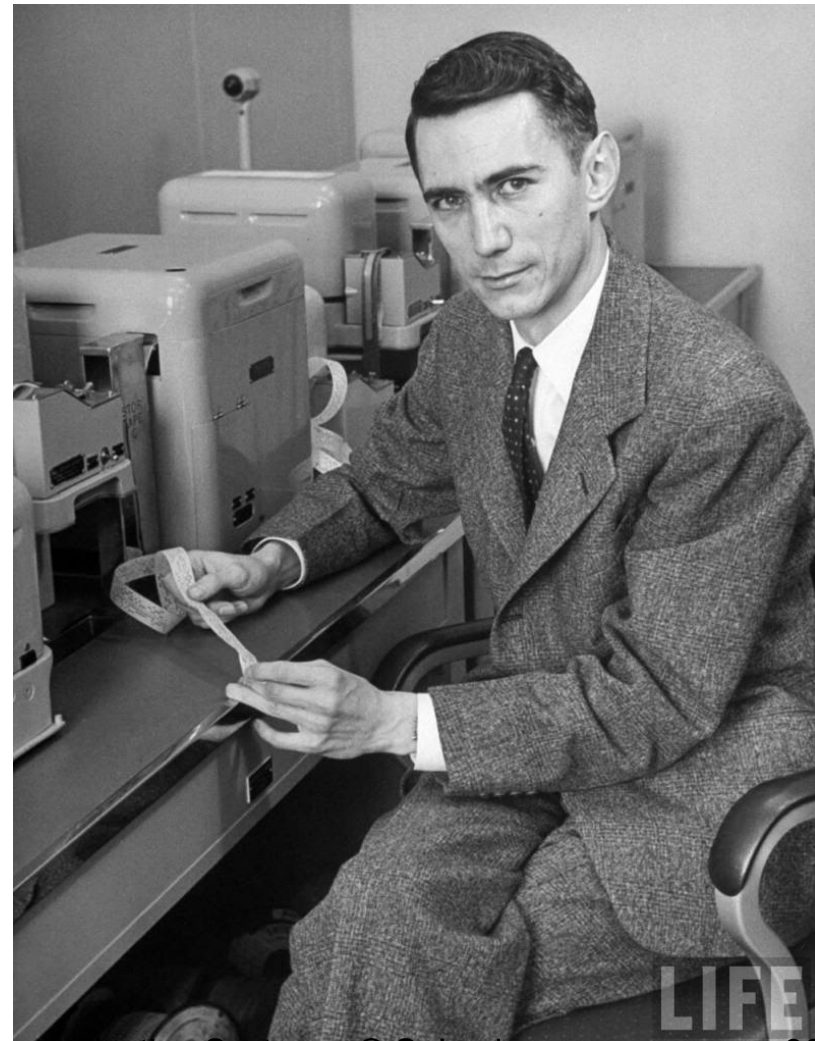
Died: 11 Feb. 2005 (aged 96), Moscow, Russia



Claude Shannon



- Farther of Information Theory
- **Born:** 30. Apr. 1916, Michigan, USA
- **Died:** 24 Feb. 2001, Massachusetts, USA



Edmund Whittaker

XVIII.—On the Functions which are represented by the Expansions of the Interpolation-Theory. By E. T. Whittaker.

(MS. received May 14, 1915. Read June 7, 1915.)

§ 1. Introduction.

Let $f(x)$ be a given function of a variable x . We shall suppose that $f(x)$ is a one-valued *analytic* function, so that its Taylor's expansion in any part of the plane of the complex variable x can be derived from its Taylor's expansion in any other part of the plane by the process of analytic continuation.

Let the values of $f(x)$ which correspond to a set of equidistant values of the argument, say $a, a+w, a-w, a+2w, a-2w, a+3w, \dots$ be denoted by $f_0, f_1, f_{-1}, f_2, f_{-2}, f_3, \dots$ etc. We shall suppose that these are all finite, even at infinity. Then denoting $(f_1 - f_0)$ by δf_0 , $(f_0 - f_{-1})$ by δf_{-1} , $(\delta f_1 - \delta f_0)$ by $\delta^2 f_0$ etc., we can write out a "table of differences" for the function; the notation which will be used will be evident from the following scheme:—

Argument.	Entry.								
$a - 2w$	f_{-2}
		δf_{-1}
$a - w$	f_{-1}	$\delta^2 f_{-1}$
		δf_{-1}	$\delta^2 f_{-1}$	$\delta^3 f_{-1}$
a	f_0	$\delta^2 f_0$	$\delta^3 f_0$	$\delta^4 f_0$
		δf_0	$\delta^2 f_0$	$\delta^3 f_0$
$a + w$	f_1	$\delta^2 f_1$
		δf_1	$\delta^2 f_1$	$\delta^3 f_1$
$a + 2w$	f_2

(1)

Now it is obvious that $f(x)$ is not the only analytic function which can give rise to the difference-table (1): for we can form a new function by adding to $f(x)$ any analytic function which vanishes for the values $a, a+w, a-w, a+2w, \dots$ of the argument, and this new function will have precisely the same difference-table as $f(x)$. All the analytic functions which give rise in this way to the same difference-table will be said to be *cotabular*. Any two cotabular functions are equal to each other when the argument has any one of the values $a, a+w, a-w, a+2w, \dots$, but they are not equal to each other in general when the argument has a value not included in this set.

Born: 24 Oct. 1873, Lancashire, England

Died: 24 Mar. 1956 (aged 82), Edinburgh, Scotland



Harry Nyquist

Certain Factors Affecting Telegraph Speed¹

By H. NYQUIST

SYNOPSIS: This paper considers two fundamental factors entering into the maximum speed of transmission of intelligence by telegraph. These factors are signal shaping and choice of codes. The first is concerned with the best wave shape to be impressed on the transmitting medium so as to permit of greater speed without undue interference either in the circuit under consideration or in those adjacent, while the latter deals with the choice of codes which will permit of transmitting a maximum amount of intelligence with a given number of signal elements.

It is shown that the wave shape depends somewhat on the type of circuit over which intelligence is to be transmitted and that for most cases the optimum wave is neither rectangular nor a half cycle sine wave as is frequently used but a wave of special form produced by sending a simple rectangular wave through a suitable network. The impedances usually associated with telegraph circuits are such as to produce a fair degree of signal shaping when a rectangular voltage wave is impressed.

Consideration of the choice of codes show that while it is desirable to use those involving more than two current values, there are limitations which prevent a large number of current values being used. A table of comparisons shows the relative speed efficiencies of various codes proposed. It is shown that no advantages result from the use of a sine wave for telegraph transmission as proposed by Squier and others² and that their arguments are based on erroneous assumptions.

SIGNAL SHAPING

SEVERAL different wave shapes will be assumed and comparison will be made between them as to:

1. Excellence of signals delivered at the distant end of the circuit, and
2. Interfering properties of the signals.

Consideration will first be given to the case where direct-current impulses are transmitted over a distortionless line, using a limited range of frequencies. Transmission over radio and carrier circuits will next be considered. It will be shown that these cases are closely related to the preceding one because of the fact that the transmitting medium in the case of either radio or carrier circuits closely approximates a distortionless line. Telegraphy over ordinary land lines

¹ Presented at the Midwinter Convention of the A. I. E. E., Philadelphia, Pa. February 4-8, 1924, and reprinted from the Journal of the A. I. E. E. Vol. 43, p. 124, 1924.

Born: 7 Feb. 1889, Värmland, Sweden
Died: 4 Apr. 1976 (aged 87), Texas, US



Sampling: if not bandlimited?

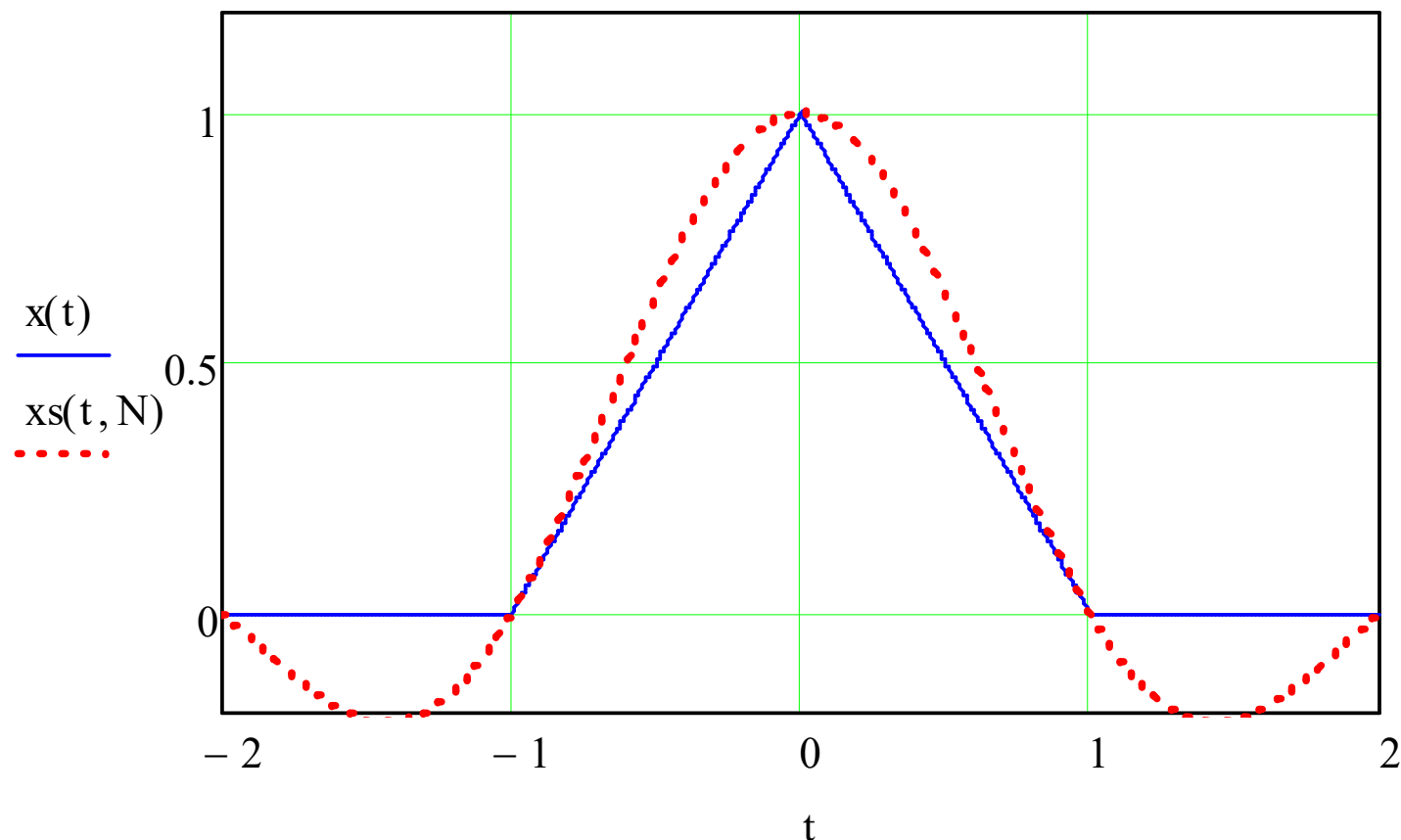
- What to do if $x(t)$ is not (ideally) bandlimited?
- Extra LPF is needed to make it bandlimited (pre-filtering)
- Approximate reconstruction only

$$\tilde{x}(t) \approx x(t)$$

- Practical: all practical signals are not ideally bandlimited
- Block diagram

Sampling $\wedge (t)$ (no pre-filtering)

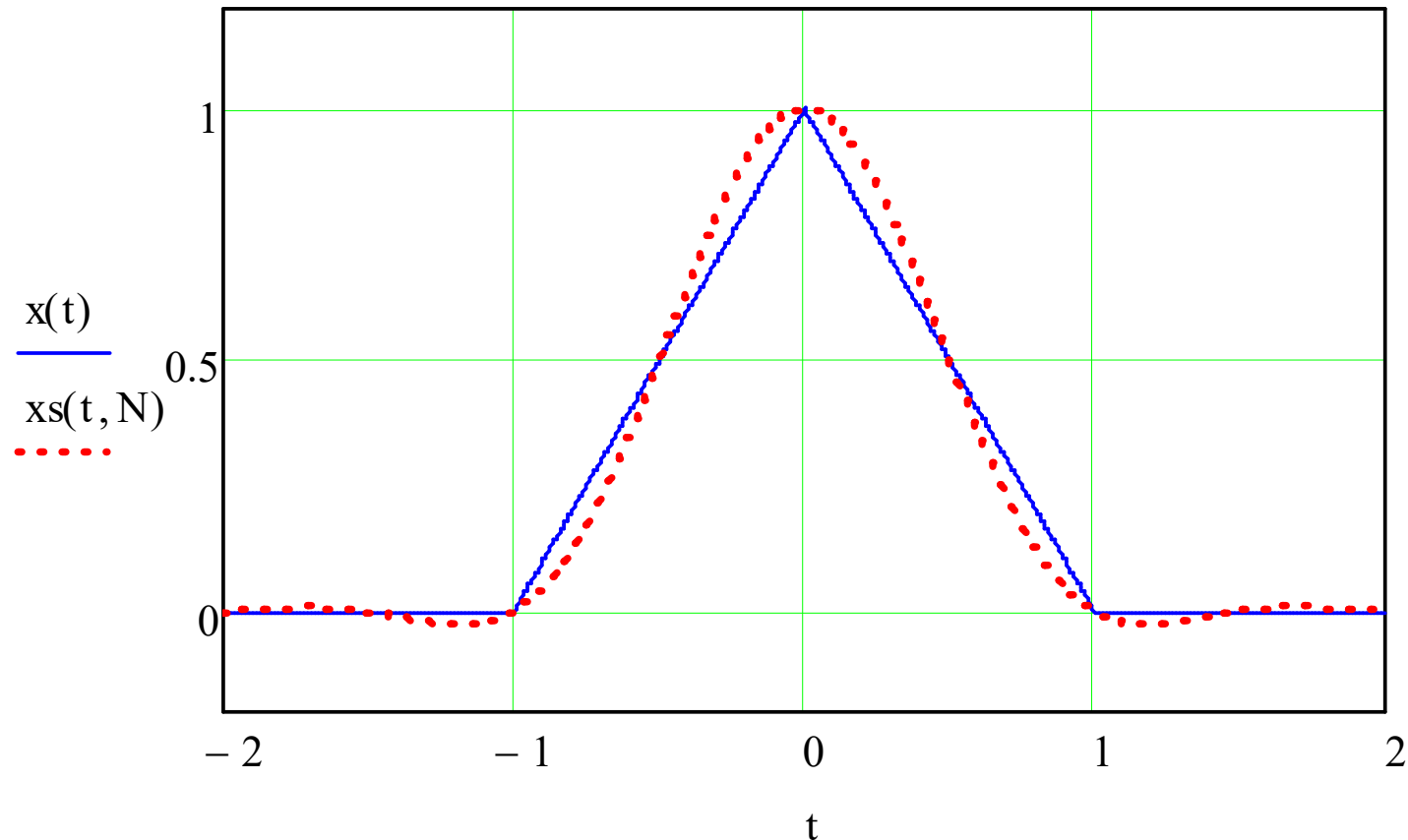
$$T_s = 1, \quad N_s = 2N + 1 = 3$$



$$x_s(t, N) = \sum_{n=-N}^N x(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right)$$

Sampling $\wedge (t)$ (no pre-filtering)

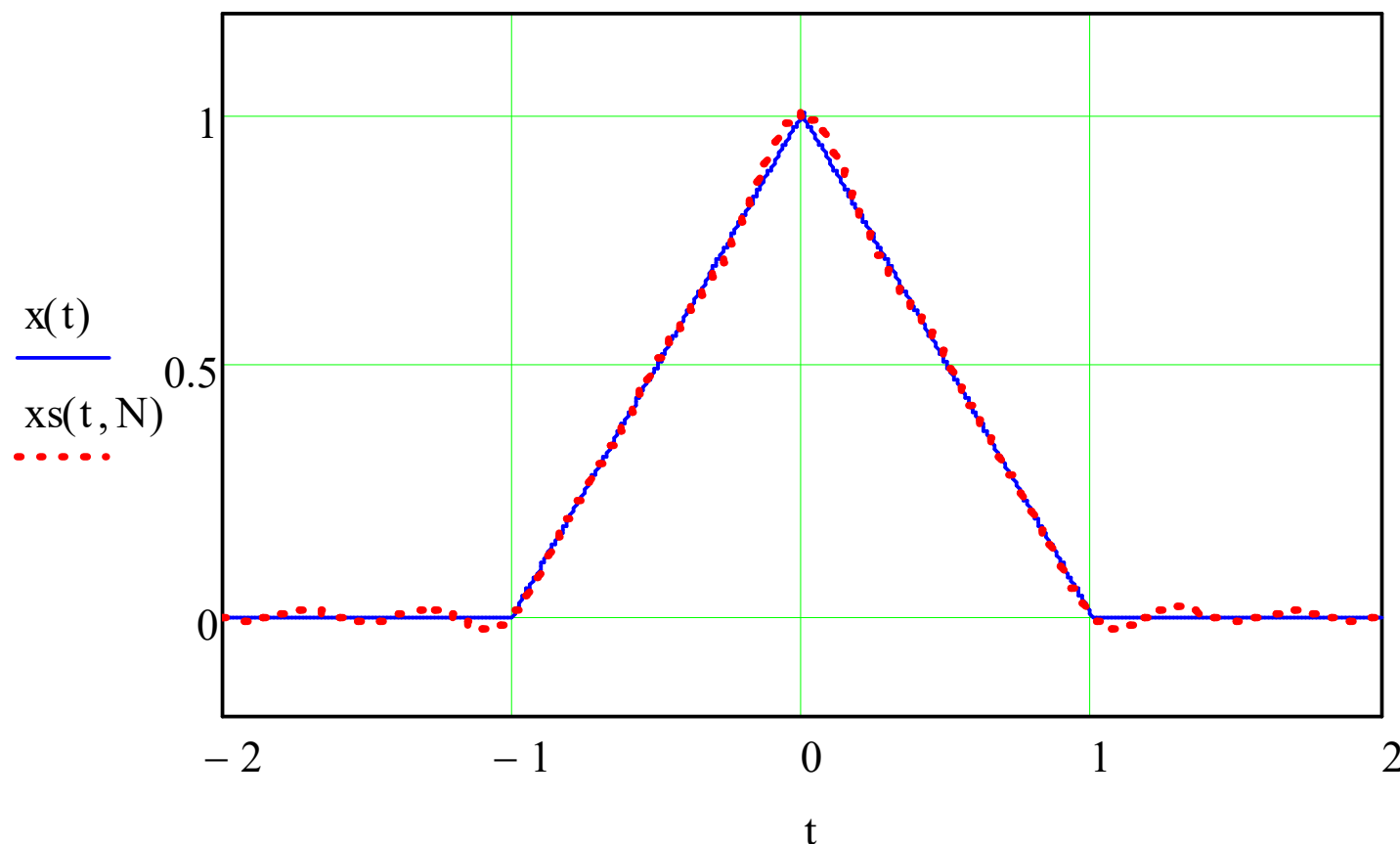
$$T_s = 1/2, \quad N_s = 2N + 1 = 5$$



$$x_s(t, N) = \sum_{n=-N}^N x(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right)$$

Sampling $\wedge (t)$ (no pre-filtering)

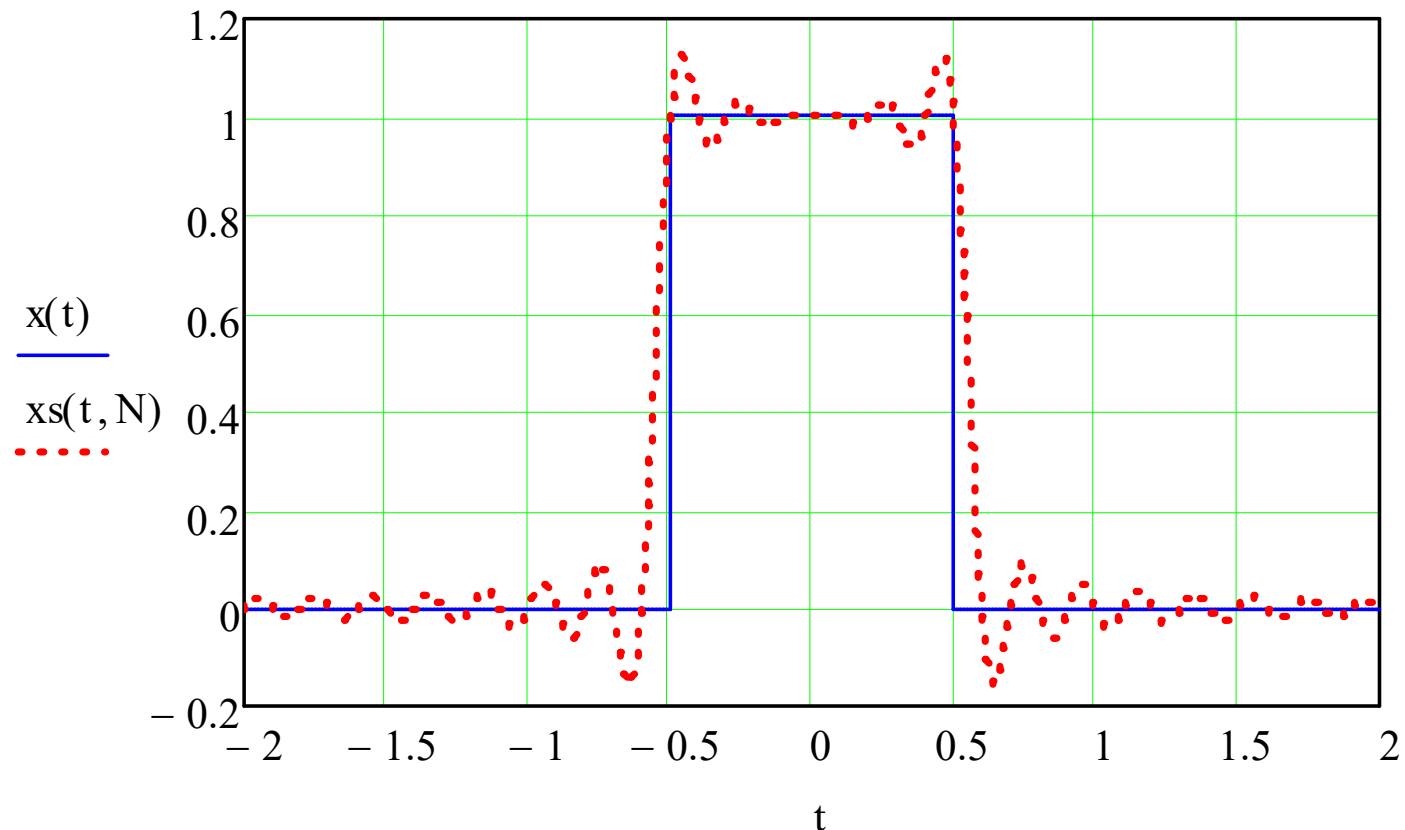
$$T_s = 1/5, \quad N_s = 2N + 1 = 11$$



Q: repeat for the rectangular pulse. How many samples are needed for accurate reconstruction ?

Sampling $\Pi(t)$ (no pre-filtering)

$$T_s = 1/10, \quad N_s = 2N + 1 = 11$$



$$x_s(t, N) = \sum_{n=-N}^N x(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right)$$

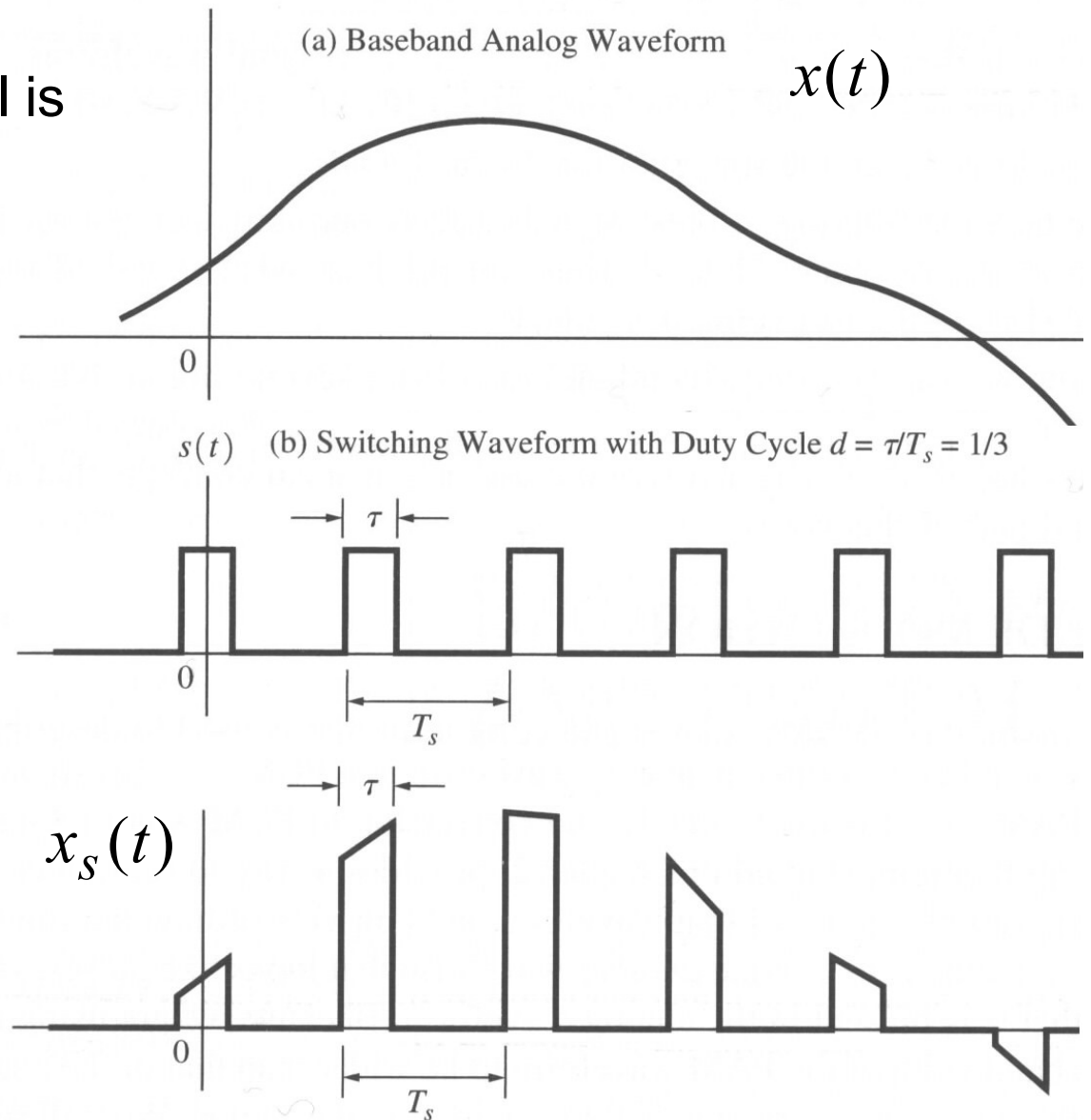
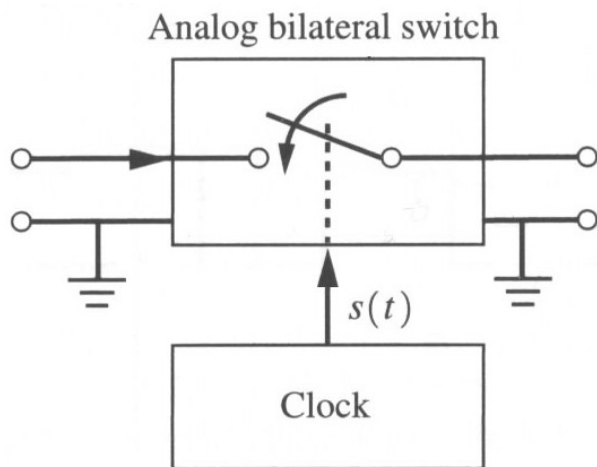
Natural Sampling (Gating)

- The sampled (PAM) signal is

$$x_s(t) = s(t)x(t),$$

$$s(t) = \sum_{k=-\infty}^{\infty} \Pi\left(\frac{t - kT_s}{\tau}\right)$$

where $f_s = 1/T_s \geq 2F_{\max}$



(c) Resulting PAM Signal (natural sampling, $d = \tau/T_s = 1/3$)

Natural Sampling (Gating): Spectrum

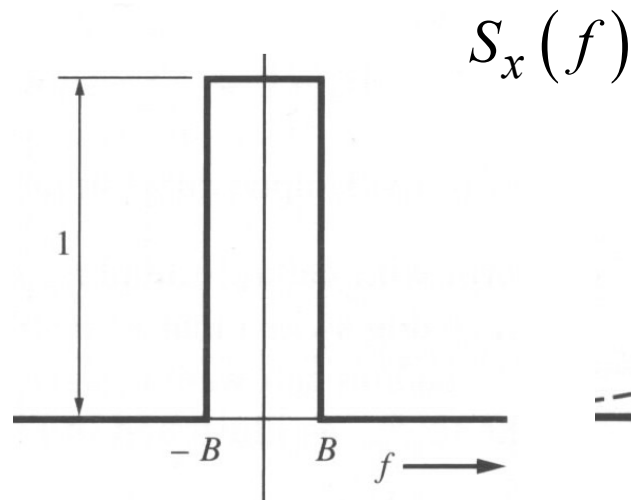
- Spectrum (FT) of the sampled (PAM) signal is

$$S_{x_s}(f) = FT[x_s(t)] = d \sum_{k=-\infty}^{\infty} \text{sinc}(kd) S_x(f - kf_s),$$

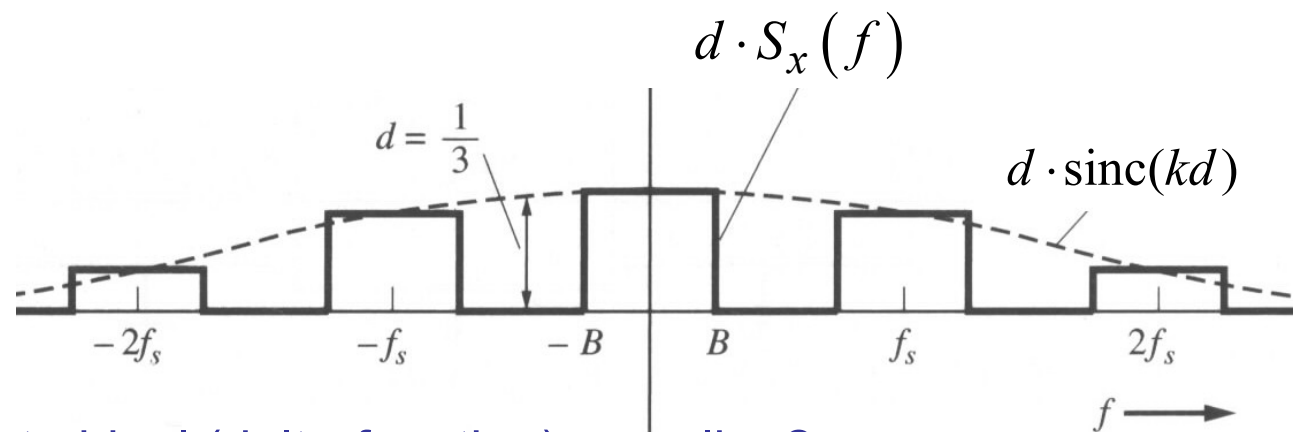
where $d = \tau/T_s$ is the duty cycle of $s(t)$.

▪ Example:

- original signal spectrum



- sampled signal spectrum $S_{x_s}(f)$



- similar to ideal (delta-function) sampling?

Natural Sampling: Proof

- Start with $x_s(t) = s(t)x(t) \leftrightarrow S_{x_s}(f) = S_x(f) * S_s(f)$
- Find Fourier series of $s(t)$:

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t}, \quad c_n = d \cdot \text{sinc}(nd)$$

- FT of $s(t)$ is $S_s(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_s)$

- Finally, $S_{x_s}(f) = S_x(f) * S_s(f) = \sum_{n=-\infty}^{\infty} c_n S_x(f - nf_s)$

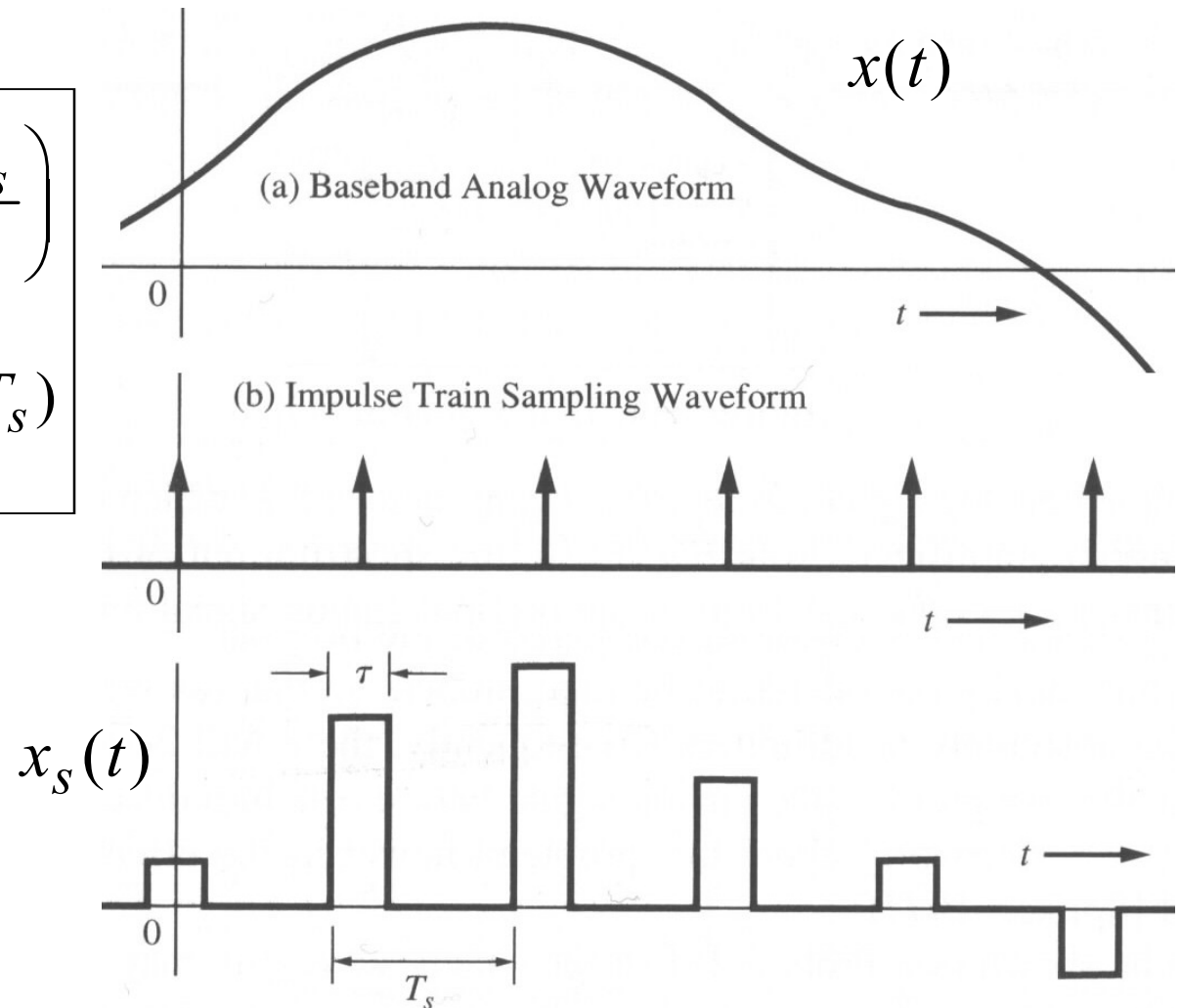
- This concludes the proof.
- How to recover (demodulate) the original signal?

Instantaneous Sampling

- Also known as flat-top PAM or sample-and-hold.
- The sampled signal is

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \Pi\left(\frac{t - kT_s}{\tau}\right)$$

$$= \Pi\left(\frac{t}{\tau}\right) * \sum_{k=-\infty}^{\infty} x(kT_s) \delta(t - kT_s)$$



(c) Resulting PAM Signal (flat-top sampling, $d = \tau/T_s = 1/3$)

References

- W. Siebert, Circuits, signals, and systems, McGraw Hill, Ch. 14.
- A.V. Oppenheim, A.S. Willsky, Signals and Systems, Ch. 7.
- B. Rimoldi, Principles of Digital Communications, Ch. 5.
- A. Lapidoth, A Foundation in Digital Communications, Ch. 8.
- H.D. Luke, The origins of the sampling theorem, IEEE Comm. Mag., vol. 37, no. 4, pp. 106-108, Apr. 1999.
- M. Unser, Sampling - 50 years after Shannon, Proc. of the IEEE, vol. 88, no. 4, pp. 569-587, Apr. 2000.
- A.J. Jerri, The Shannon sampling theorem—Its various extensions and applications: A tutorial review, Proc. of the IEEE, vol. 65, no. 11, pp. 1565-1596, Nov. 1977.

Summary

- Review of linear systems. Response of LTI system in time & frequency domains. Power transfer function.
- Signal bandwidth.
- Sampling theorem. Recovery of sampled signals.
- Good reference on filters (etc.): W. Siebert, Circuits, signals, and systems, McGraw Hill, ch. 15.
- **Homework**: Couch, 2.6, 2.7, 2.9-2.11; Oppenheim & Willsky, Ch. 2.2, 2.3, 6.1-6.4, 7.1-7.3. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand them and can solve them with the book closed.
- Do some end-of-chapter problems. Students' solution manual provides solutions for many of them.