ELG7177: MIMO Comunications

Lecture 6

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High SNR Regime

The capacity of MIMO channel is

$$C = \log|\mathbf{I} + \mathbf{W}\mathbf{R}^*| = \sum_i \log(1 + \lambda_{wi}\lambda_i^*) = \sum_i \log(\mu^{-1}\lambda_{wi}) \qquad (1)$$

and the optimal signaling (Tx) is

$$\begin{aligned} \mathbf{R}^* &= \mathbf{U}_W \mathbf{\Lambda}^* \mathbf{U}_W^+ = \sum_i \lambda_i^* \mathbf{u}_{wi}\mathbf{u}_{wi}^+ \\ \lambda_i^* &= (\mu^{-1} - \lambda_{wi}^{-1})_+ \\ \sum_i &(\mu^{-1} - \lambda_{wi}^{-1})_+ = P \end{aligned} \qquad (2)$$

high SNR: $P \to \infty$.

High SNR Regime

If $P \to \infty$, then $\mu \to 0$, so that

$$\lambda_i^* = (\mu^{-1} - \lambda_{wi}^{-1})_+ \approx \mu^{-1} = P/m \tag{3}$$

and, assuming full-rank $\mathbf{W}$,

$$\mathbf{R}^* \approx \frac{P}{m}\mathbf{I} \tag{4}$$

i.e. isotropic signaling is optimal. The capacity is

$$C \approx \log\left|\mathbf{I} + \frac{P}{m}\mathbf{W}\right| \approx \log\left|\frac{P}{m}\mathbf{W}\right| = m\log\frac{P}{m} + \log|\mathbf{W}| \tag{5}$$

(3)-(7) hold at finite SNR if

$$\mu^{-1} \gg \lambda_{wi}^{-1} \leftrightarrow P \gg m\lambda_{wm}^{-1} \tag{6}$$

High SNR Regime

The MIMO capacity is

$$C \approx m \log \frac{P}{m} + \log |\mathbf{W}| \tag{7}$$

Compare this to the SISO capacity:

$$C_1 = \log(1 + |h_{11}|^2 P) \approx \log P + \log |h_{11}|^2 \tag{8}$$

and to the Tx/Rx beamforming capacity:

$$C_b = \log(1 + \lambda_{w1} P) \approx \log P + \log \lambda_{w1} \tag{9}$$

High SNR Regime

Q1: consider free-space propagation at far field: $h_{ij} = 1$ for all i, j. Compare C, C_1, C_b.

Q2: Now assume $\mathbf{H} = \mathbf{I}$ and compare 3 capacities.

Q3: Sometimes, the high SNR regime is defined when all eigenmodes are active. Derive a condition on the SNR for this to be the case.

Q4: Find an equivalent of (4) when $\mathbf{W}$ is rank-deficient.

Multiplexing Gain

Multiplexing: launching multiple bit streams simultaneously.

Multiplexing gain r: informally,

$$C \approx r \log(1 + \alpha\gamma) \tag{10}$$

Formal definition:

$$r = \lim_{\gamma \to \infty} \frac{C}{\log \gamma} \tag{11}$$

Q1: using the WF solution, show that

$$r = \mathrm{rank}(\mathbf{H}) = \mathrm{rank}(\mathbf{W}) \tag{12}$$

Q2: when is beamforming optimal at high SNR?

Low SNR Regime

The capacity of MIMO channel is

$$C = \log |\mathbf{I} + \mathbf{W}\mathbf{R}^*| = \sum_i \log(1 + \lambda_{wi}\lambda_i^*) \tag{13}$$

The low-SNR regime is when $P \to 0$ or, more precisely, $\lambda_{wi}\lambda_i^* \ll 1$, so that

$$C \approx \operatorname{tr} \mathbf{W}\mathbf{R}^* = P\lambda_{w1} \tag{14}$$

and the optimal signaling is

$$\mathbf{R}^* \approx P\mathbf{u}_1\mathbf{u}_1^+,\ \lambda_1^* = P,\ \lambda_2^* = ... = \lambda_m^* = 0 \tag{15}$$

i.e. signaling on the largest eigenmode of $\mathbf{W}$, or beamforming along $\mathbf{u}_1$, is optimal at low SNR.

Low SNR Regime

The low-SNR MIMO capacity is

$$C \approx \operatorname{tr} \mathbf{W}\mathbf{R}^* = \lambda_{w1} P \tag{16}$$

Compare this to the SISO capacity:

$$C_1 = \log(1 + |h_{11}|^2 P) \approx |h_{11}|^2 P \tag{17}$$

and to the Tx/Rx beamforming capacity:

$$C_b = \log(1 + \lambda_{w1} P) \approx \lambda_{w1} P \tag{18}$$

Note: $\lambda_{w1} = \lambda_1(\mathbf{W}) = \sigma_1^2(\mathbf{H})$, $\mathbf{u}_1(\mathbf{W}) = \mathbf{u}_1(\mathbf{H})$.

Low SNR Regime

Q1: consider free-space propagation at far field: $h_{ij} = 1$ for all i, j. Compare C, C_1, C_b.

Q2: Now assume $\mathbf{H} = \mathbf{I}$ and compare 3 capacities.

Q3: Sometimes, the low SNR regime is defined when only one eigenmode is active. Derive a condition on the SNR for this to be the case. Which precisely eigenmode is active?

Q4: Is it possible for low and high SNR regimes to overlap, i.e. the SNR is high but only 1 eigenmode is active?

Q5: Sometimes, the low-SNR regime is called "wideband regime". Justify this definition by considering the SISO channel and letting $\Delta f \to \infty$ while the signal power P_x being fixed.

Rank-1 Channel

Let rank($\mathbf{H}$) = rank($\mathbf{W}$) = 1, so that

$$\mathbf{H} = \sigma_1 \mathbf{v}_1 \mathbf{u}_1^+, \; \mathbf{W} = \sigma_1^2 \mathbf{u}_1 \mathbf{u}_1^+ \tag{19}$$

Show that

$$\mathbf{R}^* = P \mathbf{u}_1 \mathbf{u}_1^+ \tag{20}$$

i.e. beamforming along $\mathbf{u}_1$ is optimal, at any SNR, and the capacity is

$$C = \log(1 + P\sigma_1^2) \tag{21}$$

Note that while $\mathbf{R}^*$ depends on $\mathbf{u}_1$, the capacity does not. The optimal signaling transforms the MIMO channel into equivalent SISO channel. What about optimal receiver?

In general, show that

$$\text{rank}(\mathbf{R}^*) \leq \text{rank}(\mathbf{H}) \tag{22}$$

Impact of Channel State Information (CSI)

The optimal covariance $\mathbf{R}^* = \mathbf{U}_W \boldsymbol{\Lambda}^* \mathbf{U}_W^+$ depends on the channel $\mathbf{W}$ (or $\mathbf{H}$).

How to gain the CSI?

What if the Tx/Rx does not know the channel?

Hint: consider the high-SNR regime.

Impact of Channel State Information (CSI)

CSI: via channel estimation.[1]

Incomplete/inaccurate/no CSI: compound channel model.[2][3]

The compound capacity is the largest achievable rate over a class of channels,

$$C = \max_{\mathbf{R}} \min_{\mathbf{H} \in S_H} \log |\mathbf{I} + \mathbf{H}\mathbf{R}\mathbf{H}^{+}| \qquad (23)$$

where S_H is channel uncertainty set.

[1] B. Hassibi, B. Hochwald, How much training is needed in multiple-antenna wireless links?, IEEE Trans. Inform. Theory, v. 49, pp. 951-963, Apr. 2003.

[2] S. Loyka, C.D. Charalambous, On the Compound Capacity of a Class of MIMO Channels Subject to Normed Uncertainty, IEEE Trans. Info. Theory, v.58, N.4, pp. 2048-2063, Apr. 2012.

[3] S. Loyka, C.D. Charalambous, Novel Matrix Singular Value Inequalities and Their Applications to Uncertain MIMO Channels, IEEE Trans. Info. Theory, v. 61, N. 12, pp. 6623 - 6634, Dec. 2015.

Impact of Channel State Information (CSI)

The least yet meaningful CSI:

$$S_H = \{\mathbf{H} : \operatorname{tr} \mathbf{H}^+ \mathbf{H} \geq G\} \tag{24}$$

G = minimum channel power gain.

Show that

1. isotropic signaling is optimal:

$$\mathbf{R}^* = \frac{P}{m}\mathbf{I} \tag{25}$$

2. the capacity is

$$C = \log(1 + G \cdot P/m) \tag{26}$$

3. compare this with the high-SNR regime.

Normalization of MIMO Channel Matrices

The MIMO channel model,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \boldsymbol{\xi} \tag{27}$$

The total Rx power

$$P_r = \sum_i \overline{|y_i|^2} = \overline{|\mathbf{y}|^2} = \overline{\mathbf{x}^+\mathbf{H}^+\mathbf{H}\mathbf{x}} = \mathrm{tr}\,\mathbf{W}\mathbf{R} = \frac{P}{m}\,\mathrm{tr}\,\mathbf{W} \tag{28}$$

where the last equality holds for isotropic signaling, $\mathbf{R} = \frac{P}{m}\mathbf{I}$.

The earlier-introduced SNR γ:

$$\gamma = \frac{P}{\sigma_0^2} = \frac{\text{Tx power}}{\text{Rx noise}} \tag{29}$$

Makes sense?

Normalization of MIMO Channel Matrices

A proper SNR = the Rx SNR γ_r:

$$\gamma_r = \frac{\text{Rx power}}{\text{Rx noise}} = \frac{P_r}{\sigma_0^2} = \frac{\gamma}{m} \operatorname{tr} \mathbf{W} = \gamma \tag{30}$$

where $\gamma = P/\sigma_0^2$; the last equality holds if

$$\operatorname{tr} \mathbf{W} = \operatorname{tr} \mathbf{H}^+ \mathbf{H} = \sum_{i,j} |h_{ij}|^2 = m \tag{31}$$

This is a proper channel normalization.

Also justified by the physics of antenna arrays[4].

Q: consider this for the SISO channel.

[4] S. Loyka, G. Levin, On Physically-Based Normalization of MIMO Channel Matrices, IEEE Trans. Wireless Comm., v. 8, N. 3, pp. 1107-1112, Mar. 2009.

Summary

- High/low SNR regimes
- Optimal signaling and capacity
- Multiplexing gain
- Impact of CSI, compound channel/capacity
- Normalization of MIMO channel matrices
- Multiple questions